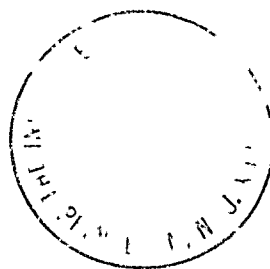


CONFERENCE ON ANALYTIC FUNCTIONS



analytic functions

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VOL - 2



Air Force Office of Scientific Research
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CONFERENCE ON ANALYTIC FUNCTIONS.

SEMINARS
ON
ANALYTIC FUNCTIONS

Volume II

Seminar III - Riemann Surfaces

Seminar IV - Theory of Automorphic Functions

Seminar V - Analytic Functions as Related
to Banach Algebras

Institute for Advanced Study
Princeton, New Jersey

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Seminar III. RIEMANN SURFACES

ABEL'S THEOREM FOR OPEN RIEMANN SURFACES

Lars V. Ahlfors

1. The theory of singular abelian differentials on open Riemann surfaces is very new and has not yet led to definitive results. The problem has been dealt with in the thesis of Bader and papers of L. Myrberg. These papers seem to open up certain possibilities, but they are also indicative of the difficulties that lie ahead.

One can agree with these authors that the main difficulty begins with the presence of infinitely many poles. However, the case of a finite number of singularities is not as innocuous as it may seem at first glance. Inevitably, in the finite case one must make the connection with the theory of differentials with a finite square norm, first developed by Nevanlinna. I propose to show that there exists in this framework, and for arbitrary Riemann surfaces, a satisfactory theory that culminates in a close analog of Abel's theorem. This is the first time, except for very special examples, that the role of Abel's theorem for the theory of open Riemann surfaces has been investigated.

2. We denote by Γ or $L^2(W)$ the Hilbert space of square integrable differentials on a Riemann surface W with the usual

definition of inner products; it is convenient to regard Γ as a complex vector space. There are two particularly important subspaces:

Γ_c , the space of closed differentials, and Γ_e , the space of exact differentials. Within the class C^1 closed and exact differentials are defined in the usual way, and the definition is extended to arbitrary differentials by forming the closure. We use notations like Γ_c^* and Γ_e^* for the subspaces formed by the conjugates of differentials in Γ_c and Γ_e respectively. It is well known that $\Gamma_h = \Gamma_c \cap \Gamma_c^*$ is the space of harmonic differentials.

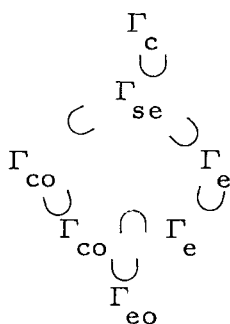
Two new subspaces Γ_{co} and Γ_{eo} are introduced by the orthogonal decomposition $\Gamma = \Gamma_c + \Gamma_{eo}^* = \Gamma_e + \Gamma_{co}^*$. Clearly, the differentials in Γ_{co} and Γ_{eo} are characterized by a certain null-behavior near the ideal boundary, but it is needless and difficult to describe this behavior in precise terms.

When the two orthogonal decompositions $\Gamma = \Gamma_c + \Gamma_{eo}^*$ and $\Gamma = \Gamma_c^* + \Gamma_{eo}$ are superimposed on each other one finds de Rham's three way decomposition $\Gamma = \Gamma_h + \Gamma_{eo} + \Gamma_{eo}^*$. It specializes to $\Gamma_c = \Gamma_h + \Gamma_{eo}$; this decomposition shows that as far as periods and boundary behavior are concerned the theory of closed differentials can be replaced by that of harmonic differentials.

I have found it very useful to introduce still another subspace.

On examining the homologies on W we find that certain cycles have the property of being homologous to a cycle that lies outside of any compact set. They are sometimes said to be weakly homologous to 0; personally, I prefer to call them dividing cycles. We shall say that a differential is semi-exact if it has zero periods along all dividing cycles, and I denote the class of semi-exact differentials by Γ_{se} . Of course, I have in mind a modified definition which makes Γ_{se} a closed subspace.

The inclusion relations between the various subspaces are indicated in the following diagram:



3. For many purposes it is sufficient to consider the intersections of these subspaces with Γ_h . For brevity we write $\Gamma_h \cap \Gamma_e = \Gamma_{he}$, $\Gamma_{co} \cap \Gamma_e = \Gamma_{ho}$, $\Gamma_h \cap \Gamma_{se} = \Gamma_{hse}$. Note that

$$\Gamma_h \cap \Gamma_{eo} = 0.$$

With these notations one has for instance the very important orthogonal decomposition $\Gamma_h = \Gamma_{ho} + \Gamma_{he}^*$. We see that Γ_{ho} coincides with Γ_h if and only if there are no non-constant harmonic functions with a finite Dirichlet-integral (the surface is of class O_{HD}).

Our definition of the class Γ_{ho} has been rather implicit, and in the general case it is not easy to recognize whether a harmonic differential belongs to this class or not. However, if W happens to be of finite connectivity, they can be spotted at once. Such a surface can indeed be considered as the interior of a compact bordered Riemann surface with a finite number of analytic contours (pointlike contours may be omitted). In this representation it is found that the differentials of class Γ_{ho} are those that vanish along the boundary.

In the case of finite connectivity it is easy to see, moreover, that Γ_{hse}^* and $\Gamma_{ho} \cap \Gamma_{he}$ are orthogonal complements. The elements of $\Gamma_{ho} \cap \Gamma_{he}$ are the differentials of harmonic functions which have a constant value on each contour. Since any such function can be expressed as a linear combination of harmonic measures in the

traditional sense I will set $\Gamma_{ho} \cap \Gamma_{he} = \Gamma_{hm}$, and I shall allow myself to refer to the elements of Γ_{hm} as harmonic measures. This notation yields the decomposition $\Gamma_h = \Gamma_{hm} + \Gamma_{hse}^*$.

What we have said applies only to the case of finite connectivity. However, it is entirely appropriate to postulate a corresponding decomposition in the general case. In other words, we define Γ_{hm} as the orthogonal complement of Γ_{hse}^* . It is then quite easy to see that $\Gamma_{hm} \subset \Gamma_{ho} \cap \Gamma_{he}$, but there is no reason to expect that these subspaces are always identical. So far, however, this question remains open.

The class Γ_{hm} plays a very important role in the subsequent study of singular differentials. It is therefore desirable to have a more direct characterization of differentials in this class. If Ω denotes a generic compact subregion of W with analytic boundary, one can show that $\Gamma_{hm}(W) = \lim_{\Omega \rightarrow W} \Gamma_{hm}(\Omega)$, in the obvious sense of norm-convergence. Since the elements of $\Gamma_{hm}(\Omega)$ are determined by linear combinations of harmonic measures in the classical sense, this is an explicit construction principle. Limitations of space do not allow me to present the proof, which is quite simple.

4. We illustrate the construction of harmonic differentials with singularities by looking at a differential of the second kind with a given double pole. Since we are using complex harmonic differentials we may assume that the singularity is of the form $\frac{dz}{(z-\zeta)^2}$ (z is a local parameter, and ζ is a particular value of z). The classical construction goes as follows:

First one constructs, in some elementary manner, a closed differential θ which has the given singularity and vanishes outside of a compact set. Because of the form of the singularity it is found that $i\theta^*$ has the same singularity as θ . Therefore, $\theta - i\theta^*$ is square integrable, and by the decomposition theorem we can write

$$\theta - i\theta^* = \omega_h + \omega_{eo} + \omega'_{eo}$$

where the notation is self-explanatory ($\omega_h \in \Gamma_h$, ω_{eo} , $\omega'_{eo} \in \Gamma_{eo}$).

It follows that

$$\omega = \theta - \omega_{eo} = i\theta^* + \omega_h + \omega'_{eo}$$

is closed and co-closed, that is to say harmonic, with the right singularity.

More generally, this construction can be used to obtain a

harmonic differential with a finite number of given singularities and a finite number of prescribed periods over non-dividing cycles. The construction of θ is always elementary, but unfortunately the final result is not unique. Indeed, we can add to θ an arbitrary exact differential which vanishes outside of a compact set. Such a differential is of class $\Gamma_{co} \cap \Gamma_e$, but not necessarily of class Γ_{eo} , and this means that ω may depend on the choice of θ .

To improve the method we decompose ω_h into $\omega_{hm} + \omega_{hse}^*$ and introduce

$$\tau = \theta - \omega_{eo} - \omega_{hm} = i\theta^* + \omega_{hse}^* + \omega'_{eo}.$$

Then τ is again harmonic with the right singularities and periods, and in addition τ^* is semi-exact. Now τ is unique. To see this, suppose that we replace θ by θ' and construct the corresponding τ' . The difference is of the form $\tau - \tau' = \theta - \theta' + \omega_{eo} + \omega_{hm}$.

Since $\theta - \theta'$ is exact and vanishes near the ideal boundary it is easy to verify, by explicit integration, that $\theta - \theta'$ is orthogonal to Γ_{hse}^* . The same is trivially true of ω_{eo} . Hence $\tau - \tau' \in \Gamma_{hm}$, but since the difference is also in Γ_{hse}^* we conclude that $\tau' = \tau$.

Consider the canonical decomposition $\tau = \phi + \bar{\psi}$ where ϕ and ψ are analytic. Because τ has analytic singularities it turns

out that ϕ inherits the singularities while ψ is singularity-free, and hence square integrable. Note that ϕ and ψ are both semi-exact.

5. Let the singular differential with double pole $(z-\zeta)^{-2}dz$ be denoted by $\tau_\zeta = \phi_\zeta + \overline{\psi_\zeta}$. It is easy to see that ψ_ζ is almost, but not quite, the reproducing kernel of Bergman. More precisely, it has the reproducing property for all semi-exact analytic differentials. ϕ_ζ bears a similar relationship to the singular kernel of Schiffer-Spencer.

Higher singularities can be obtained by differentiation with respect to ζ , simple poles and periods are generated by integration. To be specific, let c be a singular chain, and set

$$\tau(c) = \phi(c) + \overline{\psi(c)} = \int_c \phi_\zeta d\zeta + \int_c \overline{\psi_\zeta} d\zeta.$$

Then $\tau(c)$ has simple poles at the end point of c . Moreover, the periods will be determined by intersection numbers: one finds that

$$\frac{1}{2\pi i} \int_{c_2} \tau(c_1) = c_1 \times c_2.$$

What is more, one proves that $\tau(c)$ is canonically associated with

these singularities and periods in the sense of our unique construction. Let us also note that the reproducing property is expressed by the formula

$$(*) \quad (a, \psi(c)) = 2\pi \int_c a$$

which is valid for all semi-exact analytic a .

What happens if c is a cycle? Then $\phi(c)$ and $\psi(c)$ are both regular. Since the periods of $\tau(c)$ are multiples of $2\pi i$ we find that $\operatorname{Re} \tau(c)$ is exact. But this implies, by the uniqueness of our construction, that $\operatorname{Re} \tau(c) = 0$. This is possible only so that $\phi(c) = -\psi(c)$, and thus $\tau(c) = -2i \operatorname{Im} \psi(c)$. Let us introduce one more notation: $\chi(c) = \frac{1}{2\pi i} \tau(c) = -\frac{1}{\pi} \operatorname{Im} \psi(c)$. Then $\chi(c)$ has integral periods; precisely, the period of $\chi(c)$ along a cycle c' equals $c \times c'$.

If c is a dividing cycle we deduce that $\chi(c)$ is exact, and therefore zero. The important thing is that the converse holds as well. Let $\{a_i, b_j\}$ be a canonical homology basis modulo the dividing cycles. An arbitrary cycle c can be represented as a finite sum $\sum (x_i a_i + y_j b_j)$, again modulo the dividing cycles. It follows that $\chi(c) = \sum (x_i \chi(a_i) + y_j \chi(b_j))$, and if $\chi(c) = 0$ it is easily deduced that all the x_i and y_j are zero. Hence $\chi(c) = 0$ implies

indeed that c is a dividing cycle.

Finally, the reproducing property implies

$$(\alpha, \chi(c)^*) = \int_c \alpha$$

for all $\alpha \in \Gamma_{ase}$ (= analytic and semi-exact). In other words, $\chi(c)$ serves to compute the periods of all semi-exact analytic differentials.

6. Before we come to Abel's theorem, let me restate the characteristic properties of the differential τ that we have associated with a given set of singularities and periods:

1) τ is harmonic except for a finite number of isolated singularities,

2) τ^* is semi-exact,

3) outside of a compact set τ can be represented in the form $\omega_{hm} + \omega_{eo}$.

We shall say that a differential with these properties is canonical.

We have shown -- and this is the reason for introducing the whole concept -- that a canonical differential is completely determined by its singularities and periods.

We shall carry over the notation $\chi(c) = -\frac{1}{\pi} \operatorname{Im} \psi(c)$ to arbitrary chains c . If c happens to be a cycle we know that $\chi(c)$ has integral periods; what is more, we have also $\chi(c) = \frac{1}{2\pi i} \tau(c)$, so that $\chi(c)$ is canonical. It follows that we can write $\chi(c)$ as a finite linear combination of $\chi(a_i)$, $\chi(b_j)$. Differentials with such a representation will be called modular.

We propose to answer the following question: For what chains c is $\chi(c)$ modular? Clearly, in view of what we have already shown the answer depends only on the boundary ∂c .

Suppose first that $\chi(c)$ is modular. We can write $\phi(c) + \psi(c) = \tau(c) + \psi(c) - \overline{\psi}(c) = \tau(c) - 2\pi i \chi(c)$. Hence $\phi(c) + \psi(c)$ is canonical, and its periods are multiples of $2\pi i$. Because of the latter property we can set $\phi(c) + \psi(c) = d \log f$, where f is a single-valued meromorphic function with a finite number of zeros and poles. It is seen that the zeros and poles correspond precisely to the coefficients of ∂c . In other words, ∂c is the divisor of f .

We shall say that f is quasi-rational if and only if $d \log f$ is canonical. This implies that f has only a finite number of zeros and poles, equally many of each.

Let us now start from a quasi-rational function f . It is

clear that we can construct a $\tau(c)$ which has exactly the same singularities and periods as $d \log f$. This implies $d \log f = \tau(c)$
 $= \phi(c) + \psi(c) + 2\pi i \chi(c)$, and we deduce that $\chi(c)$ is analytic. But since $\chi(c)$ is real it must be zero. We have found a chain c with boundary equal to the divisor of f which is such that $\chi(c) = 0$. It follows that $\chi(c')$ is modular for any other chain c' with the same boundary.

ABEL'S THEOREM. The differential $\chi(c)$ is modular if and only if ∂c is the divisor of a quasi-rational function.

There is an equivalent formulation which has a more familiar ring. If $\chi(c)$ is modular we have seen that c can be replaced by another chain with the same boundary, such that $\chi(c)$ is identically zero. Therefore, we obtain:

A 0-chain is the divisor of a quasi-rational function if and only if it is the boundary of a singular 1-chain c with the property that

$$\int_c \alpha = 0$$

for all semi-exact square integrable analytic differentials α .

Indeed, by formula (*) the condition is equivalent to $\psi(c) = 0$,
and hence to $\chi(c) = 0$.

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DETERMINATION OF AN AUTOMORPHIC FUNCTION FOR A GIVEN ANALYTIC EQUIVALENCE RELATION

Léonce Fourès

1. INTRODUCTION

Let $f(z)$ be meromorphic on a Riemann surface R .

$f(\zeta_1) = f(\zeta_2)$ defines a local analytic relation between points in neighborhoods of z_1 and z_2 where $f(z_1) = f(z_2)$. This relation may also be written $\zeta_2 = f^{-1} \cdot f(\zeta_1) = \phi(\zeta_1)$ and may be extended to an analytic but not single-valued application of R onto itself.

Such a function ϕ is called an automorphism function for f or briefly an automorphism for f .

An arbitrary analytic application of R onto itself is, in general, not an automorphism function. Automorphisms for meromorphic functions have been studied by Shimizu, who determined also the functions for which all the automorphisms are linear. This determination has been obtained by using Nevanlinna's proof of a theorem of Bloch, and it can also be obtained by using parabolic regularly ramified coverings of the plane. Important results have also been obtained by G. af Haellstroem who studied very deeply the cases f meromorphic and f algebraic.

If ϕ is an automorphism for f , this is also true for all positive and negative iterates of ϕ ; if ψ is another automorphism,

any function of the form $\psi \cdot \phi$ is also an automorphism for f . It may happen that the product $\psi \cdot \phi$ is not well defined: $\psi \cdot \phi$ may split into several automorphism functions for f . If, for instance, ϕ is an automorphism, ϕ^{-1} is again an automorphism and the same is true for $\phi^{-1} \cdot \phi$; the identity belongs to the set of functions of the form $\phi^{-1} \cdot \phi$, but usually ϕ is neither single-valued nor schlicht and $\phi^{-1} \cdot \phi$ can be split into several functions.

2. THE EQUIVALENCE RELATION

Analytic equivalence relation. Instead of starting with functions ϕ_i , let us consider an equivalence relation on R ; since f is single-valued, the relation $f(x) = f(y)$ is an equivalence relation \equiv on R which satisfies

- I. The intersection of any compact set on R with any coset of this relation is a finite set;
- II. If $y_0 \equiv x_0$ there exists between the local parameters of neighborhoods of x_0 and y_0 an analytic relation $y = \phi(x)$ which associates equivalent points.

DETERMINATION OF AN AUTOMORPHIC FUNCTION

One should remark that the analytic relation $y = \phi(x)$ is necessarily of the form

$$(1) \quad y = G[a_0 x^{\frac{p}{q}} (1 + g(x))]$$

where $G(0) = 0$, $G'(0) = 1$, and $g(0) = 0$ (G and g are single-valued).

We have been able to establish that conditions I and II are sufficient for the existence of a function f single-valued on R and assuming equal values at equivalent points. In the present paper I intend to give an explicit construction of such a function f by generalizing Poincaré's theory of theta series.

This existence theorem enables us to obtain an interpretation of the special form (1) in terms of iteration. Let us suppose that instead of the form (1) we have between local parameters of neighborhoods of x_0 and y_0 a general relation of the form

$$y = \phi(x) = a_0 x^{\frac{p}{q}} (1 + a_1 x^{\frac{1}{q}} + \dots + a_n x^{\frac{n}{q}} + \dots);$$

then in the neighborhood of y_0 there are q points corresponding to x by ϕ .

ϕ^{-1} associates p points in the neighborhood of x_0 to each

point in the neighborhood of y_0 . We get $q(p-1) + 1$ equivalent points in the neighborhood of x_0 ; then applying ϕ again we get $q(p-1)(q-1) + q$ equivalent points near y_0 , and by repeating this process one may get an infinity of equivalent points in the neighborhood of x_0 , and this contradicts I. On the other hand, if $\phi(x)$ is of the form (1) there is a stabilization between ϕ and ϕ^{-1} and one gets only p equivalent points in the vicinity of x_0 , q points in the vicinity of y_0 . One should note that there may be more than p equivalent points near x_0 .

The order p of the equivalence relation at x_0 may be defined as the maximum number of equivalent points in an arbitrarily small neighborhood of x_0 . Then in any neighborhood of x_0 there exists a particular neighborhood, and a local uniformizing parameter such that the ratio of the parameters of two equivalent points is a p -th root of unity. It follows that the functions G and g in (1) for the points x_0 and y_1 are not independent of those for x_0 and y_0 .

Regular equivalence relation. Let us suppose we have an analytic equivalence relation in the plane P (or a connected part of it) and that this relation satisfies I and II.

DETERMINATION OF AN AUTOMORPHIC FUNCTION

Let us denote by $A(x)$ the set of points which are equivalent to x (we call $A(x)$ the orbit of x). $A(V)$ will be the set of points for which an equivalent is in V . $A(x)$ will be called an algebraic orbit if it contains at least one point of order greater than 1. $A(x)$ will be transcendental if for any neighborhood $V(x)$ one of the components of $A(V(x))$ is non-compact.

Let Σ' be obtained by removing all the algebraic and transcendental orbits from P , and let Σ be a compact region in $\Sigma' - A(0)$.

V will be called a normal region if there are not two equivalent points in V . One can prove that every function ϕ_i which associates equivalent points to points in V maps V conformally and 1-1 onto a region V_i with the property $V_i \cap V_j = \emptyset$ for $i \neq j$.

3. THE FUNCTION $F(x)$

Let us suppose first that P is unbounded. We intend to study the behavior of the series

$$\sum_i \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2}.$$

If V is a normal region

$$\int_V \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2} d\omega = \frac{1}{4} \cdot \text{area of } V_i \text{ on the sphere}$$

from which it follows that

$$(2) \quad \sum_i \int_V \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2} d\omega = \frac{1}{4} \cdot K < \pi$$

where $K = \text{area of } A(V) \text{ on the sphere.}$

One can easily prove

LEMMA 1. On every compact region V of area σ , there exists at least one point at which

$$\sum_i \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2} \text{ is convergent and } < \frac{\pi}{\sigma}.$$

With the help of a finite covering of Σ by circles C_k of radius ρ , such that the circles concentric with these and of radius 2ρ are normal regions, one can show

$$\text{LEMMA 2. } \left| \frac{\phi'_i}{\phi_i} \right| < \frac{4}{\rho} \text{ for every } x \text{ in } \Sigma.$$

It is then possible to construct a square net over Σ with the sides of the squares less than $\frac{1}{16\sqrt{2}} \rho$. In every square there exists by

Lemma 1 a point ξ_j on which the series

DETERMINATION OF AN AUTOMORPHIC FUNCTION

$$\sum_i \frac{|\phi'_i(\xi_j)|^2}{(1+|\phi_i(\xi_j)|^2)^2} \text{ is convergent.}$$

Then for every x belonging to the same square as ξ_j one has

$$\frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2} < 16 \frac{|\phi'_i(\xi_j)|^2}{(1+|\phi_i(\xi_j)|^2)^2}.$$

We obtain

THEOREM 1. The series $\sum_i \frac{\phi'_i(x)^2}{\phi_i(x)^4}$ converges

absolutely and uniformly in Σ where it defines a holomorphic function $F(x)$.

This theorem has to be completed for the neighborhoods of points belonging to $A(0)$. For such a point and its vicinity there is only one exceptional value $i(\ell)$ for which Lemma 2 does not hold: this value is such that $\phi_{i(\ell)}(x_\ell) = 0$. From this remark one may obtain:

COMPLEMENT TO THEOREM 1. $F(x)$ is also defined in a neighborhood of any point in $A(0)$ and is meromorphic in Σ' .

To establish Theorem 1 one has to show that $F(x)$ is single-valued. This will be done by showing that a continuation of $F(x)$ along a loop L permutes the functions ϕ_i in the series defining $F(x)$; one shows that by such a continuation one does not lose any ϕ_i and one does not find the same ϕ_i twice; this is done by considering the continuation along the inverse loop L^{-1} .

If P (and hence Σ') is bounded, we may suppose Σ' in the unit circle. One may remark that

$$\int_V |\phi'_i(x)|^2 d\omega = \text{plane area of } V_i.$$

One gets a lemma similar to Lemma 1 but with $\sum_i |\phi'_i(x)|^2$ replacing

$\sum_i \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2}$. One can construct a square net as in the

unbounded case and deformation theorems give $|\phi'_i(x)| < \lambda |\phi'_i(\xi)|$

where λ depends only on the side of the squares. The series

$\sum \phi'_i(x)^2$ is then uniformly convergent and in the following takes

the place of $\sum_i \frac{\phi'_i(x)^2}{\phi_i(x)^4}$.

DETERMINATION OF AN AUTOMORPHIC FUNCTION

4. THE FUNCTION $\theta(x)$

It is then possible to follow Poincaré's method. Let $R(t)$ be a rational function of t with poles a_1, a_2, \dots, a_p in Σ' and with finite behavior at infinity. All the equivalents of a_1, \dots, a_p are in Σ' . Let then V_0, V_1, \dots, V_p be normal neighborhoods of $0, a_1, \dots, a_p$ respectively. $R(t)$ is bounded in $\Sigma' - \bigcup A(V_k) = \Sigma^*$ and

$$\theta(x) = \sum_i R(\phi_i) \frac{\phi_i'(x)^2}{\phi_i(x)^4}$$

is then meromorphic in Σ' , holomorphic in Σ^* . $\theta(x)$ is bounded in Σ^* and certainly has poles in Σ' and therefore is not constant.

Let then x and y be two equivalent points in Σ' .

$$\theta(x) = \sum_i R(x_i) \frac{\left(\frac{dx_i}{dx}\right)^2}{x_i^4}$$

$$\theta(y) = \sum_i R(x_i) \frac{\left(\frac{dx_i}{dy}\right)^2}{x_i^4} = \theta(x) \left(\frac{dx}{dy}\right)^2$$

it follows that if R_1 is another rational function with at least one pole distinct from those of R then (for the correspondingly defined function θ_1)

$$\frac{\theta(x)}{\theta_1(x)} = \frac{\theta(y)}{\theta_1(y)} \text{ in } \Sigma^* \text{ and also in } \Sigma'.$$

The function $f(x) = \frac{\theta(x)}{\theta_1(x)}$ assumes equal values at all points belonging to the same orbit. Furthermore $f(x)$ is not constant since $\theta(x)$ and $\theta_1(x)$ do not have the same poles.

5. ALGEBRAIC ORBITS

Let us suppose that all points belonging to the same orbit have the same order n (where n depends on the orbit).

In general the above defined $f(x)$ has a transcendental singularity at every point of an algebraic orbit, since nothing ensures that $f(x)$ cannot assume the same value on an infinity of different orbits. One should remark that the special structure of multiple points did not play any role and the function $f(x)$ would be the same even if infinitely many points of an orbit would cluster at a multiple point. One cannot hope to prove boundedness of the previously defined $f(x)$ in a neighborhood of a multiple point.

A normal neighborhood of a multiple point will be conformally mapped onto a circle, the equivalence relation being transformed into a rotation.

One can show that all functions ϕ_i are still schlicht and all

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proofs for the regular case still hold, but one has to replace inequality (2) by

$$(2') \quad \sum_i \int_V \frac{|\phi'_i(x)|^2}{(1+|\phi_i(x)|^2)^2} d\omega = \frac{1}{4} n \cdot K < n\pi$$

where K = area of $A(V)$ on the sphere.

Furthermore n has a maximum on Σ since Σ is compact. One should remark that, except for the case $n = 2$, $\sum_i \frac{\phi'_i(x)^2}{\phi_i(x)^4} = 0$ when x belongs to an algebraic orbit; indeed, the set $\{\phi_i\}$ can be split into several subsets, all the functions contained in any particular subset assuming the same value at x . It follows that the derivatives of any two functions within the same subset differ only by a constant factor, which is an n -th root of unity. However, absolute and uniform convergence of the series in Theorem 1 must be deduced from inequalities (2') together with deformation theorems which apply only to schlicht functions.

General Case. Let there be given on R an equivalence relation which satisfies I and II. One may suppose there are no transcendental orbits by removing them if necessary from the initial Riemann surface. I and II imply boundedness of the order

$n(x_{k_i})$ for every algebraic orbit $A_k(x_{k_i} \in A_k)$. Then there exists a number v_k (for each algebraic orbit) and a covering (R^*, ψ) of R which is regularly ramified of order $\frac{v_k}{n(x_{k_i})}$ at the point x_{k_i} .

One can define an analytic equivalence relation \equiv on R^* :
 $x \equiv y$ if $\psi(x) \equiv \psi(y)$, and this relation satisfies I and II. One may use the complex plane for R^* : every function f^* assuming equal values at equivalent points of R^* defines a function $f(x)$ assuming equal values at equivalent points of R . f^* has already been constructed since v_k is the common order of all points of an algebraic orbit on R^* .

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A THEOREM CONCERNING THE EXISTENCE OF DEFORMABLE CONFORMAL MAPS

Maurice Heins

1. We shall say that a Riemann surface F admits deformable conformal maps into itself (or belongs to the class \mathcal{D}) provided that there exists a continuous map f from $F \times [0,1]$ into F satisfying the two conditions: (a) for each $t \in [0,1]$, $f_t: p \rightarrow f(p,t)$ is a (directly) conformal map (not necessarily univalent) of F into itself, (b) $f_0 \neq f_1$.

A problem which presents itself at once in the study of the homotopy of conformal maps is to characterize the Riemann surfaces of class \mathcal{D} . It is easy to see that the sphere, the finite plane, the punctured plane and the tori all belong to the class \mathcal{D} . There remain to be considered the Riemann surfaces with hyperbolic universal covering surface. These may be divided into two classes, one being the class of hyperbolic Riemann surfaces, the other the class of Riemann surfaces which themselves are not hyperbolic but which have hyperbolic universal covering surface. No surface of this latter class can belong to \mathcal{D} . The case of compact Riemann surfaces of genus ≥ 2 is classical. There are only a finite number of conformal maps of such a surface into itself. The validity of the

assertion for the case of non-compact surfaces is established in [3, §§6, 7]. Thus there remain for consideration only the hyperbolic Riemann surfaces.

In this paper we shall establish the following theorem:

A hyperbolic Riemann surface belongs to the class \mathcal{B} if and only if it admits non-constant bounded analytic functions.

That the existence of non-constant bounded analytic functions is a sufficient condition for a Riemann surface to belong to the class \mathcal{B} is of course trivial. We shall therefore confine our attention to showing that this condition is necessary for hyperbolic surfaces.

The proof will be based on our prior studies on the Lindelöf principle [1] and Lindelöfian maps [2]. These papers will be referred to henceforth as LP and LM respectively.

2. We recall [LM] that a non-constant meromorphic function ϕ whose domain is a hyperbolic Riemann surface F is termed Lindelöfian provided that

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$$\sum_{\phi(r)=w} n(r) \mathcal{G}_F(p, r) < +\infty$$

for $p \in F$, $w \in$ extended plane, $\phi(p) \neq w$. Here \mathcal{G}_F is the Green's function of F and $n(p)$ is the multiplicity of ϕ at p .

As a first step toward establishing the desired result, we prove the following lemma:

Let F and G denote hyperbolic Riemann surfaces.

Suppose that there exists a continuous map f from $F \times [0, 1]$ into G satisfying (i) $f_t: p \rightarrow f(p, t)$ is a (directly) conformal map of F into G for each $t \in [0, 1]$, (ii) $f_0 \neq f_1$. Then F admits a Lindelöfian meromorphic function.

Proof: Let $q \in F$ be such that $f_0(q) \neq f_1(q)$. Let \mathcal{G}_G denote the Green's function of G . We introduce

$$h(p, t) \equiv \mathcal{G}_G[f(p, t), f_0(q)], \quad (p, t) \in F \times [0, 1].$$

Let γ denote a singular 1-cycle of F whose carrier omits the antecedents of $f_0(q)$ with respect to f_0 and f_1 . Then the period of the conjugate of $h(p, 0)$ along γ is congruent (mod 2π) to the period of the conjugate of $h(p, 1)$ along γ . Consequently, there

exists a non-constant meromorphic function ϕ on F satisfying

$$\log |\phi(p)| \equiv h(p, 1) - h(p, 0).$$

By LM, §5b we conclude that ϕ is Lindelöfian.

3. Now a Lindelöfian meromorphic function is not of type B1 in the extended plane [LM, §6]. Consequently the closure of the image of ϕ contains a subset of positive capacity at each point of which ϕ is not of type B1 [LP, §17]. It follows that there exist three harmonic functions u_k ($k = 1, 2, 3$) on F satisfying:

(i) $0 < u_k < 1$, $k = 1, 2, 3$, (ii) G.H.M. $\min(u_k, u_j) = 0$, $k \neq j$. Here

"G.H.M." denotes "greatest harmonic minorant".

We see readily that at least one of u_2, u_3 , say u_2 , does not admit a representation of the form

$$au_1 + \beta$$

where a and β are constants.

4. We turn now to the consideration of a hyperbolic Riemann surface F of class \mathcal{B} and suppose that $f(p, t)$ is an

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admitted continuous family of conformal maps of F into itself. We consider in the present context functions u_1 and u_2 of §3.

If either u_1 or u_2 is the real part of an analytic function on F , it is trivial that F admits non-constant bounded analytic functions. We put this case aside.

It will be convenient to introduce the abelian differential δv generated from a harmonic function v defined on a Riemann surface. Specifically δv is the abelian differential given in the parameter neighborhood associated with a local uniformizer τ by

$$[(v \circ \tau)_x - i(v \circ \tau)_y] \quad (z = x + iy).$$

We observe that $\delta u_2 / \delta u_1$ is not constant. If this were not the case and $\delta u_2 / \delta u_1 = \alpha + i\beta$ (α, β real), then from $\delta(u_2 - u_1) = i\beta \delta u_1$ we would conclude either that u_2 is linear in u_1 ($\beta = 0$) or else that the periods of δu_1 are both real and pure imaginary and hence that u_1 is the real part of an analytic function on F ($\beta \neq 0$). Both of these possibilities are excluded.

Further we note that

$$\left(\frac{\delta u_2}{\delta u_1} \right) \circ g = \frac{\delta(u_2 \circ g)}{\delta(u_1 \circ g)}$$

where g is a conformal map into F .

We assert that for some (k_o, t_o) , $k_o = 1, 2$, $0 < t_o \leq 1$,

$u_{k_o} \circ f_{t_o} - u_{k_o} \circ f_o$ is not constant. It will then follow that there

exists on F a non-constant analytic function with bounded real part

since the periods of the conjugate of $u_{k_o} \circ f_t$ are independent of t ,

$0 \leq t \leq 1$, and hence that F admits a non-constant bounded analytic function.

If this assertion were not true, we would conclude that

$$\frac{\delta(u_2 \circ f_t)}{\delta(u_1 \circ f_t)} = \left(\frac{\delta u_2}{\delta u_1} \right) \circ f_t$$

is independent of t , and hence that f_t is independent of t . But

$f_o \neq f_1$. The contradiction is manifest.

* * *

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THE EXISTENCE OF DEFORMABLE CONFORMAL MAPS

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A DIRECT CONSTRUCTION OF ABELIAN DIFFERENTIALS

ON RIEMANN SURFACES

Albert Pfluger

The following theorem is a classical one: On a compact Riemann surface there exists an Abelian integral of the first kind, and up to a constant only one, whose periods have prescribed real parts. An Abelian integral is the integral function of a holomorphic differential $\alpha = \omega + i*\omega$, where ω is a harmonic form and $*\omega$ its adjoint form. Let g be the genus of the closed surface R and let C_1, \dots, C_{2g} be a homology basis of 1-cycles for R . The above theorem is equivalent to the following: If π_1, \dots, π_{2g} are arbitrarily given real numbers, there exists exactly one harmonic form ω on R which satisfies the conditions

$$\int_{C_\nu} \omega = \pi_\nu, \quad \nu = 1, \dots, 2g.$$

Since a total harmonic form vanishes identically, ω is uniquely determined by the periods π_ν .

One way to prove the existence is by normalized differentials of the third kind and is not a direct way. Another method uses Dirichlet's principle. It starts with a minimal sequence, whose existence is quite simply proved, but no construction of it is given. The purpose of this paper is to give a direct construction of a harmonic form with given periods.

Let ω be a real closed linear differential form on R , that is a linear form $\omega = a dx + b dy$, where a and b are real functions of the local uniformizer $z = x + iy$ and satisfy the condition $b_x = a_y$. Let Ω be the linear space of these forms ω . It is easy to construct a closed ω with prescribed periods. Two forms in Ω are called cohomologous, if they have the same periods. The problem is to construct the harmonic form which is cohomologous to an arbitrary given $\omega \in \Omega$.

For this purpose we introduce in Ω the inner product

$$(\omega_1, \omega_2) = \int_R (a_1 a_2 + b_1 b_2) dx dy,$$

$\omega_i = a_i dx + b_i dy$, $i = 1, 2$, whereby Ω becomes an inner product space.

Let ω_0 be an element in Ω and put

$$d = \inf_{(\omega_0)} ||\omega||,$$

where ω runs through the cohomology class of ω_0 . There exists a sequence ω_n , $n = 1, 2, \dots$ of elements in this cohomology class (ω_0) , so that $\lim_{n \rightarrow \infty} ||\omega_n|| = d$. Dirichlet's principle asserts that there exists a ϕ in (ω_0) with $||\phi|| = d$, and this ϕ will then be harmonic. In the following we will give a construction of such a minimal sequence.

We choose a finite covering of the closed surface R by 2-cells V_0, \dots, V_k , that is by simply connected domains bounded by analytic curves. The union of these finitely many curves is designated by A . Let $[4]$ be the linear space of all real continuous functions f on R , which are continuously differentiable in $R - A$ and whose Dirichlet-Integral $D_{R-A}(f)$ is finite. We call

$$(\omega_0) = \{\omega \mid \omega = \omega_0 + df, f \in [f]\}$$

the "cohomology class" of ω_0 . To every V_i we associate an operator ϕ_i , $i = 0, 1, \dots, k$, in the following way. To a $\omega \in (\omega_0)$ there exists a continuous function f_i on V_i with $df_i = \omega$ in $V_i - A$. We construct the harmonic function F_i in V_i , which is continuous on V_i and equals f_i on the boundary of V_i . This can be done by Poisson's integral, because V_i is conformally equivalent with the circle. For any $\omega \in (\omega_0)$ we define $\phi_i \omega$ by

$$\phi_i \omega = \begin{cases} \omega & \text{in } R - V_i \\ dF_i & \text{in } V_i \end{cases} \quad i = 0, 1, \dots, k.$$

ϕ_i is obviously linear and $\phi_i \omega$ again belongs to (ω_0) .

THE CONSTRUCTION: Starting with ω_0 we put

$\phi_i \omega_n = \omega_{n+1}$, $n = 0, 1, \dots$, where i runs permanently through the numbers $0, 1, \dots, k$ alternately ascending and descending.

$$\begin{aligned} \phi_i \omega_0 &= \omega_1, \phi_2 \omega_1 = \omega_2, \dots, \phi_k \omega_{k-1} = \omega_k, \\ \phi_{k-1} \omega_k &= \omega_{k+1}, \dots, \phi_1 \omega_{2k-2} = \omega_{2k-1}, \phi_0 \omega_{2k-1} = \omega_{2k}, \end{aligned}$$

and so on:

$$(1) \quad \phi_i \omega_{2k\nu+i-1} = \omega_{2k\nu+i}, \quad i = 0, 1, 2, \dots, k, \quad \nu = 1, 2, \dots$$

We assert that the ω_n converge to a harmonic form ϕ in (ω_0) .

To prove this statement we need the following

LEMMA 1: Let ω and ω' be two forms in (ω_0) and let ω' be harmonic in V_i . Then we have

$$(2) \quad (\omega', \omega - \phi_i \omega) = 0.$$

PROOF: Since $\omega - \phi_i \omega = 0$ in $R - V_i$, (2) is equivalent to $(\omega', \omega - \phi_i \omega)_{V_i} = 0$, this inner product being taken with respect to V_i . But in V_i we have $\omega = df$, $\phi_i \omega = dF$ and $\omega' = du$, where $f = F$ on the boundary of V_i , $D(F) \leq D(f) < \infty$, u harmonic with $D(u) < \infty$ in V_i . It will be easy to find a sequence of continuous and piecewise

smooth functions g_n with compact support in V_i , so that

$\lim_{n \rightarrow \infty} D(f - F - g_n) = 0$. From $D(u, g_n) = 0$ it follows

$$D(u, f - F) = (\omega', \omega - \phi_i \omega)_{V_i} = 0.$$

Going back to the sequence ω_n in (1) we take two positive integers m and n with $m \equiv \pm n \pmod{2k}$, that is $m = 2k\mu \pm i$, $n = 2k\nu \pm i$, $i = 0, 1, 2, \dots, k; \mu, \nu = 1, 2, \dots$. From $\omega_n = \phi_i \omega_{n-1}$ and Lemma 1 it follows $(\omega_m, \omega_{n-1} - \omega_n) = 0$, since ω_m is harmonic in V_i :

$$(3) \quad (\omega_m, \omega_n - \omega_{n-1}) = 0, \quad m \equiv \pm n \pmod{2k}.$$

From this property we conclude that $\{\omega_n\}$ is a Cauchy sequence.

But to do this we have no need to use the special nature of the ω and we can work in an abstract inner product space. The result, which will be proved later, is the following:

LEMMA 2: Let Ω be a real inner product space and $\{\omega_n\}$, $n = 1, 2, \dots$ a sequence of elements in Ω with the property (3). Then $\{\omega_n\}$ is a Cauchy sequence.

To construct the harmonic form ϕ , to which the $\{\omega_n\}$ are to converge, we take a fixed cell V_i and put $\omega_{\nu}^{i \pm i} = \omega_{2k\nu \pm i}$,

$\nu = 0, 1, 2, \dots$. By restriction to V_i we have $\lim_{\mu, \nu \rightarrow \infty} \|\omega_\nu^i - \omega_\mu^i\|_{V_i} = 0$.

The differentials ω_ν^i are harmonic in V_i and form a Cauchy sequence in the Hilbert space $H(V_i)$ defined by the square integrable harmonic differentials in V_i . We conclude that the sequence ω_ν^i converges to a harmonic form ϕ_i in V_i : $\lim_{\nu \rightarrow \infty} \|\omega_\nu^i - \phi_i\|_{V_i} = 0$. Because $\{\omega_n\}$ is a Cauchy sequence we also have

$$(4) \quad \lim_{n \rightarrow \infty} \|\omega_n - \phi_i\|_{V_i} = 0, \quad i = 0, 1, \dots, k.$$

Now let V_i and V_j be two overlapping cells. By restriction to the intersection $V_i \cap V_j$, (4) gives

$$\lim \|\omega_n - \phi_i\|_{V_i \cap V_j} = 0 \quad \text{and} \quad \lim \|\omega_n - \phi_j\|_{V_i \cap V_j} = 0$$

which leads to $\|\phi_i - \phi_j\|_{V_i \cap V_j} = 0$ or $\phi_i = \phi_j$ in $V_i \cap V_j$. Thus, ϕ_i and ϕ_j are harmonic continuations of each other and the forms ϕ_i join together into a single harmonic form ϕ on R . From

$$\lim_{n \rightarrow \infty} \|\omega_n - \phi\|_{V_i} = 0, \quad i = 1, 2, \dots, k, \quad \text{we conclude}$$

$$(5) \quad \lim_{n \rightarrow \infty} \|\omega_n - \phi\|_R = 0,$$

which proves the statement given above. But we do not know from this whether ϕ belongs to the cohomology class (ω_0) . To prove it

does, we observe that all ω_n belong to (ω_0) . Furthermore, to each 1-cycle C on R there exists in Ω a form θ_C with

$$(6) \quad \int_C \omega = (\theta_C, \omega), \quad \omega \in \Omega.$$

The ω_n , $n = 0, 1, 2, \dots$ having all the same periods, we find by (6) $(\theta_C, \omega_n - \omega_0) = 0$. From this and (5) we conclude $(\theta_C, \phi - \omega_0)$ which by (6) leads to $\int_C \phi = \int_C \omega_0$ for all 1-cycles C , that is: ϕ and ω_0 have the same periods.

PROOF OF LEMMA 2: Put $m = n$ in (3). Then we get

$$||\omega_n - \omega_{n-1}||^2 = ||\omega_{n-1}||^2 - ||\omega_n||^2. \quad \text{Thus the norms } ||\omega_n||$$

form a non-increasing, and therefore convergent, sequence:

$$(7) \quad \lim_{n \rightarrow \infty} ||\omega_n|| = d.$$

Hence

$$\lim_{n \rightarrow \infty} ||\omega_n - \omega_{n-1}|| = 0$$

and by induction, for every fixed positive integer N :

$$(8) \quad \lim_{n \rightarrow \infty} ||\omega_{n+N} - \omega_n|| = 0.$$

If $m \equiv -n \pmod{2k}$, we have $n-1 \equiv -(m+1) \pmod{2k}$ too. From the equation (3) and the corresponding one with $n-1$ instead of m and

$m+1$ instead of n : $(\omega_{n-1}, \omega_m - \omega_{m+1}) = 0$, we get $(\omega_m, \omega_n) = (\omega_{m+1}, \omega_{n-1})$ and by induction $(\omega_{m+2k\nu}, \omega_m) = \|\omega_{m+k\nu}\|^2$. From this we conclude $\|\omega_{m+2k\nu} - \omega_m\|^2 = \|\omega_{m+2k\nu}\|^2 + \|\omega_m\|^2 - 2\|\omega_{m+k\nu}\|^2$ and by (7)

$$\lim_{m, \nu \rightarrow \infty} \|\omega_{m+2k\nu} - \omega_m\| = 0, \quad m = 2kq.$$

This leads together with (8) to

$$\lim_{m, n \rightarrow \infty} \|\omega_m - \omega_n\| = 0,$$

which proves the Lemma. The given construction of harmonic forms works well for compact Riemann surfaces, but the question arises what happens for non-compact ones. We know by a theorem of Behnke and Stein that to any closed form ω_0 on a Riemann surface R there exists a harmonic one with the same periods as ω_0 , but more than one, if R is non-compact. Even in the case that the norm

$$\|\omega\| = \int_R (a^2 + b^2) dx dy, \quad \omega = a dx + b dy,$$

is finite, a harmonic form is not determined by its periods generally. But to every closed form ω_0 on R , with finite norm, there exists one and only one harmonic form ω with finite norm, such that ω and ω_0 have the same periods and $\omega - \omega_0$ is orthogonal to

the differentials of the continuous and continuously differentiable functions with compact support. The question is, whether the foregoing method can be used to construct such a harmonic form. For this purpose let us start with a closed form ω_0 of finite norm and a covering of the Riemann surface by 2-cells $V_0, V_1, \dots, V_m, \dots$, which has to be infinite in the non-compact case. Similarly as above we put $\omega_{n+1} = \phi_{i_n} \omega_n$, $n = 0, 1, 2, \dots$, where now i runs through the numbers $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, \dots$. Will we obtain a Cauchy sequence again? I cannot prove it because periodicity plays an important role in Lemma 2. But a less direct way is the following one. Let k be any positive number. With respect to the cells V_0, V_1, \dots, V_k we construct a form ϕ_k in exactly the same manner as in the compact case. ϕ_k is harmonic in the union of the V_0, V_1, \dots, V_k and equals ω_0 elsewhere. As ϕ_k is orthogonal to $\omega_0 - \phi_k$ one proves, following usual patterns, that the sequence $\{\phi_k\}$, $k = 1, 2, \dots$, converges to a harmonic form ϕ in the sense that $\|\phi - \phi_k\| \rightarrow 0$. This form satisfies all conditions given above.

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THE FIRST VARIATIONS OF THE DOUGLAS FUNCTIONAL
AND THE PERIODS OF THE ABELIAN INTEGRALS
OF THE FIRST KIND *

H. E. Rauch

1. In the course of some investigations (Rauch [2], [3], [4]) on the problem of the moduli I encountered a formula connecting the quantities in the title which seems novel and of possible value in various problems in conformal mapping and Riemannian function theory, although any such applications must await discussion on another occasion. The first person to discover connections among the quantities in question was J. Douglas [1] in his solution of the general Plateau problem, but his use of a special representation leaves the relationships buried implicitly, whereas I have made the relationship more explicit and much simpler by the use of the mechanism of the quadratic differential and its attendant apparatus as first conceived by Teichmueller [5].

2. Let S_z and S_w be two Riemann surfaces of the same genus on which z and w denote respectively generic local

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parameters. Let ϕ be a homeomorphism of S_z on S_w which is continuously differentiable everywhere with the possible exception of a finite number of points. Let $\lambda dw d\bar{w}$ be a conformal metric on S_w . By means of ϕ one may write on S_z :

$$1) \quad \lambda dw d\bar{w} = E dx^2 + 2F dx dy + G dy^2 = b dz d\bar{z} + 2 \operatorname{Re}(a dz^2)$$

where $z = x + iy$, $b = (E + G)/2$ and $a = (E - G)/2 - iF$.

Douglas's functional for the map ϕ is defined as

$$2) \quad J(\phi) = \iint_{S_z} \left(\frac{E+G}{2} \right) dx dy \\ = \frac{i}{2} \iint_{S_z} b dz d\bar{z}$$

where the exterior product is used under the integral sign.

It is now necessary to consider what constitutes a variation.

An infinitesimal variation of the map ϕ of S_z on S_w can be considered to be an infinitesimal transformation of S_z on itself followed by ϕ . In each parameter such a transformation will be given by $z \rightarrow z' = z + \epsilon \omega^z$ where ϵ is small, ω^z is differentiable in z and \bar{z} and $\omega^z/dz = \omega^w/dw$ in overlapping parameters z and w . However, it is essential also to consider variations of the surface S_z obtained as follows: put a new conformal metric on S_z defined

by

$$3) \quad \lambda' dz' d\bar{z}' = b dz d\bar{z} + 2\varepsilon \operatorname{Re}(q dz^2)$$

where $q dz^2$ is an everywhere finite (but as yet not necessarily analytic) quadratic differential and the new set of parameters z' is defined by putting the right side of 3) in the form of the left side.

Now in order to calculate the first variation of $J(\phi)$ under the indicated changes of map ϕ and structure of S_z one must express the latter type variation in the form $z \longrightarrow z + \varepsilon\omega$, too, since in the present crude calculus that is the only way one can proceed.

Rewriting 3) as

$$4) \quad \lambda' dz' d\bar{z}' = b dz d\bar{z} \left\{ 1 + 2\varepsilon \operatorname{Re} \left(\frac{q}{b} \frac{dz^2}{|dz|^2} \right) \right\}$$

setting $z' = z + \varepsilon\omega$ and comparing first powers of ε in 4) one finds

$$5) \quad \bar{\omega} = q/b.$$

However, the solution of 5) cannot be a tensor on S_z unless the

change of type given by 3) be the trivial one obtained by an infinitesimal transformation of S_z onto itself. 5) is easily solved by an integral formula analogous to the solutions of Poisson's equation (Teichmueller [5], p. 81) if one first uniformizes the surface by means of, say, the Grenzkreis uniformization, in which case z will be the uniformizing variable. The solution will then be unique up to analytic functions in z .

3. For such variation one can easily compute the first variation of $J(\phi)$ (Rauch [3]) and find it to be

$$6) \quad \delta J = 2\varepsilon \operatorname{Im} \iint_{S_z} a \omega \frac{dz}{z} d\bar{z} + \frac{i\varepsilon}{2} \iint_{S_z} d\{b(\omega d\bar{z} - \bar{\omega} dz)\}.$$

One can then, by Stokes' formula (d being the exterior differential), transform the last term to an integral over the boundary Γ of a fundamental domain (Γ may be thought of in intrinsic terms as the boundary of a canonical dissection of S_z).

Suppose now that $q dz^2$ is the particular analytic quadratic differential $d\zeta_i d\zeta_j$, $d\zeta_i$ being the i -th normal Abelian differential of the first kind on S_z . Then except for the factor $1/2\pi i$ the first double integral in 6) is precisely $\delta' \bar{\pi}_{ij}$, π_{ij} being the j -th

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period of $d\zeta_i$ and δ' being the variation obtained by using a in
5) instead of q (Rauch [2]), so that there is this peculiar reci-
procity:

$$7) \quad \delta J = 4\pi \operatorname{Re} \delta' \pi_{ij} + \frac{i\epsilon}{2} \int_{\Gamma} b(\omega d\bar{z} - \bar{\omega} dz).$$

On the other hand if one now assumes that ϕ is a critical
map for $J(\phi)$ then as one easily sees from 6) (Rauch [3]) $a dz^2$ is
an everywhere finite analytic quadratic differential and hence a
linear combination $\sum c_{ij} d\zeta_i d\zeta_j$ of basis differentials (for a non-
hyperelliptic surface); and a similar reasoning now establishes

$$8) \quad \delta J = 4\pi \operatorname{Re} \sum c_{ij} \delta \pi_{ij} + \text{boundary term}$$

for the same variation δ , which is the formula mentioned in the
introduction.

The detailed discussion of the boundary term and its uses
and implications will not be given here.

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STABILITY PROBLEMS ON BOUNDARY COMPONENTS

Leo Sario

1. In my lecture at the Helsinki Colloquium [13] I indicated a new proof for Radó's theorem that every 2-dimensional manifold with conformal structure possesses a countable basis of open sets. I also indicated how extremal and existence problems on Riemann surfaces can be solved without making use of the countability axiom. This approach was made possible by the introduction into complex analysis of directed limits by Ahlfors. Here I shall attack a particular extremal problem. It can, as a consequence, be considered without utilizing countability.

For preparation, we shall need an existence theorem.

Existence questions often amount to the construction of a harmonic function with given singularities and given behavior near the boundary. The classical case is that of a closed Riemann surface W , where a point β and a punctured neighborhood G of β with boundary α are given. On G , a harmonic function s is prescribed with a singularity at β . The problem is to construct a harmonic function p on W with singularity s at β , i.e., such that $p - s$ belongs to the family H of harmonic functions h on G that possess a harmonic continuation to β . The problem is known

to have a solution if and only if the flux of s vanishes, $\int_{\alpha} ds^* = 0$.

In the general case of an open Riemann surface W of finite or infinite genus, the point β is replaced by the (ideal) boundary β of W , the punctured neighborhood G by a neighborhood G of the boundary β , and the function s by a harmonic function s on G with an arbitrary behavior as one approaches β . The problem is that of constructing, on all of W , a harmonic function p which has the behavior s in G . This means that $p - s$ belongs to the class H of harmonic functions h on G which, in a sense, have no singularities on β . More precisely, h has no source on β , i.e., its flux vanishes. Furthermore, h is to be bounded on G by its extrema on the relative boundary α of G . We say that h is associated with its values f on α by an operator L , and we may write these conditions as follows:

$$1^{\circ}. \quad \int_{\alpha} d(Lf)^* = 0$$

$$2^{\circ}. \quad \min_{\alpha} f \leq Lf \leq \max_{\alpha} f.$$

We call a linear operator that satisfies these conditions (and gives a

unique Lf for any given f) a normal linear operator. The following result then is known [8].

THEOREM 1. Given G , L , and s , the condition

$$\int_a ds^* = 0 \text{ is necessary and sufficient for the existence}$$

of a harmonic function p on W such that $p - s$ belongs to H on G .

The theorem is quite general. First, G need not be connected but may consist of a (disconnected) boundary neighborhood plus some disjoint open sets interior to W . The functions s can prescribe the behavior in the boundary neighborhood and also some singularities in the other open sets constituting G . The operator L may be chosen to be different in different components of G . Thus we have obtained a harmonic function s on W with given singularities inside W and given behavior as one approaches various parts of the boundary.

In particular, L may be chosen to be the operator L_0 that gives, roughly speaking, a vanishing normal derivative to $h_0 = L_0 f$ on β ; or else the operator L_1 that gives constant values to $h_1 = L_1 f$ on the various boundary components [11]. Then the

corresponding principal functions p_0 and p_1 on W give standard domain functions and mappings onto canonical regions as special cases of the theorem. Also, linear combinations $hp_0 + kp_1$ furnish solutions to a variety of extremal problems concerning Taylor coefficients, boundary integrals, areas of the maps, and related functionals [10].

2. To illustrate the use of Theorem 1, let us construct Abelian integrals of the first kind with a prescribed period.

Suppose the period 1 is required along a given 1-cycle δ^* on W . We take the conjugate cycle δ and divide it into the arcs δ_1 and δ_2 by the points z_1 and z_2 . Then we embed δ_1 into a simply-connected region $D_1 \subset W$ and set $G_1 = D_1 - \gamma_1$. For a second component G_2 of G we take any neighborhood of the ideal boundary β of G . We map D_1 conformally onto a disk D_1 and choose

$$s_1 = \frac{1}{2\pi} \arg \frac{z - z_1}{z - z_2}$$

to be the singularity function in G_1 . In G_2 we simply take $s_2 \equiv 0$. The condition $\int ds^* (= \sum_i \int ds_i^*) = 0$ is then clearly satisfied. The

linear operator L in G_1 is chosen to give harmonic functions on G_1 with harmonic extensions to D_1 . On G_2 any L can be used. Our theorem then provides us with a harmonic p' on $W - \delta_1$ with the jump 1 across δ_1 . In a similar manner, by first embedding δ_2 in a simply-connected region, we obtain a harmonic p'' on $W - \delta_2$ with the jump 1 across δ_2 . The summation $p' + p''$ cancels the singularities, and we have the desired integral with period 1 along δ^* .

If, in addition, one chooses L for the boundary neighborhood G_2 of β to be L_0 or L_1 , then $p' + p''$ has finite norm (Dirichlet integral), and the existence of differentials with finite norm has been established on arbitrary open Riemann surfaces (cf. [7]).

3. After these general remarks we proceed to our actual problem.

It is well known that the complement of a point γ with respect to the Riemann sphere cannot be conformally mapped onto the complement of a disk. The reason is, roughly speaking, that an infinitesimal disk near γ would be compressed in the radial direc-

tion, stretched in the circular direction and this would destroy conformality. On the other hand, consider a totally disconnected closed plane point set β whose complement W with respect to the plane has finite area. Then we can make use of the relation $O_{SB} = O_{SD}$ by Ahlfors and Beurling [2], where O_{SB} and O_{SD} signify the classes of Riemann surfaces which do not admit analytic univalent functions that are bounded or possess a finite Dirichlet integral, respectively. The identity mapping of W has a finite Dirichlet integral, and we conclude that W is mapped, by some function SB , onto a bounded region. After taking inversions in both planes, the picture looks like this: In the original plane the point at infinity has become the origin, and the other components accumulate to it with such density that, even after the inversion, the complementary area remains finite. In the image plane γ has become a circle approached from the outside by other components. The plausible reasoning in this case is that the compressing and stretching is now taken care of by these other components, and an infinitesimal disk in W near γ can remain undistorted.

This might lead one to believe that such a heavy accumulation of other components is always necessary if γ is to be stretched

into a continuum. A very simple example shows that this, however, is not so. Take a circle γ' approached from the outside by an infinite sequence of concentric circles with decreasingly small gaps on the right. The complementary area W can be mapped onto a horizontal slit region, and for reasons of symmetry we may choose the slits on the real axis, proceeding to the right with decreasing length. Because of the inaccessibility of γ' , its "image" must reduce to a single point γ , the accumulation point of the slits. Here we have a rather "lonesome" point γ which, however, hides in itself the power of a proper continuum.

The natural question now arises: How can we know whether a given boundary component γ of a region W is always a point, under all univalent mappings of W ? Further, if a continuum is given, under what circumstances is it always a continuum? And finally, when is a given component sometimes a point, sometimes a continuum?

It seems convenient to use the following suggestive terminology [12]. A boundary component γ is weak, strong or unstable, according to whether it is always a point, always a continuum, or sometimes a point, sometimes a continuum. The problem is to

determine to which category a given γ belongs. The problem is very difficult. I can only report some very modest preliminary results.

4. We begin by deriving a sufficient condition for γ to be strong. For this we need an extremal theorem.

Consider a plane region V bounded by a Jordan curve γ_V and a closed set δ encircled by γ_V and such that $\gamma_V \cap \delta$ is void. Suppose the origin $z = 0$ is in V . We are interested in the class $\{F\}$ of univalent analytic functions $F(z)$ on V with the normalization $F(0) = 0$, $F'(0) = 1$ and such that the image under F of γ_V is a circle with radius $r(F)$, say. Denote by $F_0(z)$ the function with the additional property that the "image" δ_{F_0} under $F_0(z)$ of δ has vanishing area and that each component of δ_{F_0} is a radial slit (or a point). The existence of $F_0(z)$ is uniquely given by Theorem 1, for $\log|F_0(z)|$ appears as p if s is chosen to be $\log|z|$ near $z = 0$, the harmonic measure of γ_V with flux 2π near γ_V , and identically zero near δ , while L is taken as L_1 near γ_V , L_0 near δ .

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We designate the logarithmic area of the image δ_F under F of δ by

$$A = - \int_{\delta} \log |F(z)| d \arg F(z).$$

Then the following theorem can be established (the proofs of results reported in this lecture will appear in [14]):

THEOREM 2. The radial slit mapping F_0 has the following extremal property:

$$\max_{F \in \{F\}} [2\pi \log r(F) + A(F)] = 2\pi \log r_0(F).$$

One might suggest that since the area $A(F)$ encircled by δ_F appears in the functional to be maximized, a greater value would be obtained by widening the radial slits so as to contain some area. But as we do this, the radius $r(F)$ shrinks, and we lose more than we gain.

5. Before continuing, we wish to make a side remark. The radius $r(F)$ usually varies as F ranges in $\{F\}$. It is of interest to know for what kinds of δ the radius is rigid, i. e. the same for

all F . A complete solution of this problem was found by a student of mine, J. Seewerker, and I would like to quote his result here [16]:

THEOREM 3. W has rigid radius if and only if

$$\delta' \in O_{AD}.$$

Here δ' is the complement of δ with respect to the sphere and O_{AD} signifies the class of Riemann surfaces that do not possess non-constant analytic functions with a finite Dirichlet integral [5, 6]. The theorem can be proved as a consequence of Theorem 2, and was established by Seewerker using other methods.

6. We return to our problem. A sufficient condition for a boundary component γ of a plane region W to be weak can now easily be established. We may assume that γ is the outer contour if W is bounded, or that it contains (or reduces to) the point at infinity if W is unbounded.

Let V be a subregion of W bounded by a Jordan curve $\gamma_V \subset W$ and by the subset δ of the boundary of W that is encircled by γ_V . The idea now is the following. To make sure γ is strong, we try

to give γ its weakest possible representation. If γ is a continuum in this weakest form, it must be so always, and it is strong. The weakest representation is obtained by pushing γ as far as possible towards the point at infinity, and Theorem 2 tells us this is achieved for γ_V by the mapping F_{oV} of V onto a radially slit disk. We set

$$d = \lim_{V \rightarrow W} r(F_{oV})$$

and state:

THEOREM 4. γ is strong if $d < \infty$.

The proof is based on Theorem 2. I suspect the converse is true as well, i. e., $d = \infty$ implies γ is strong.

7. For weak components, somewhat more complete results have been obtained. We start with an analogue of Theorem 2, and denote by $F_1(z)$ the function in $\{F\}$ that maps V into a disk such that the image δ_{F_1} of δ has vanishing area and each component of δ_{F_1} is a circular slit (or a point).

THEOREM 5. The circular slit disk mapping $F_1(z)$ has the extremal property $\min_{F \in \{F\}} [2\pi \log r(F) - A] = 2\pi \log r(F_1)$.

This result, incidentally, was simultaneously and independently found, by use of other methods, by H. Wittich.

A reasoning parallel to that in §6 now leads to the following statement, where

$$c = \lim_{V \rightarrow W} r(F_{1V}).$$

THEOREM 6. γ is weak if $c = \infty$.

The condition is also necessary. The proof is based on Theorem 5. It may be of interest to note that the constant c here is the negative of the logarithm of the capacity of γ as defined in [9]. Theorem 6 could, therefore, also be proved as a consequence of the equality $C_\gamma = O_{SB}$ [l. c.], where C_γ is the class of Riemann surfaces for which each boundary component has vanishing capacity.

8. Now that we have reduced the question of the weakness of γ to the evaluation of the measure c , the question arises to what extent methods used in other type problems could be employed here. In [5], a modular criterion was given for O_{AD} . The analogue in the present case is as follows. Suppose $\{D_n\}$ is a sequence of disjoint doubly-connected subregions of W tending to a given boundary component in the sense that D_n separates D_{n+1} (and γ) from a fixed $\zeta \in W$. Map D_n conformally onto a circular annulus with radii 1 and $\mu_{n\gamma}$ (> 1).

THEOREM 7. γ is weak if $\prod \mu_{n\gamma} = \infty$.

The proof was given by a student of mine, N. Savage, in [15].

A number of methods originally developed for the classical type problem were adapted in [5] to the case O_{AD} . It was shown that these methods amounted to the evaluation of the moduli of doubly-connected regions forming certain sequences. Consequently, the same methods can be used in the present case [15].

In Nevanlinna's method of regular parametric disks [4], the following conditions are imposed. Any two intersecting parametric disks $|z_\nu| < 1$ and $|z_\mu| < 1$ are to overlap sufficiently, in the

sense that the intersection meets both disks $|z_\nu| < 1 - d$ and $|z_\mu| < 1 - d$, where d is independent of the disks. Moreover, no point of W is to belong to more than a fixed number N of the disks. Suppose the annular regions D_n in the modular criterion above are replaced by doubly-connected unions of Nevanlinna disks, and that ℓ_n is the number of disks constituting D_n . Then it can be shown that γ is weak if $\sum \frac{1}{\ell_n}$ diverges.

In Ahlfors' method of conformal metric [1] one takes a fixed point $\zeta \in W$ and considers the set $S(\rho)$ of curves consisting of points at distance ρ from ζ in a given conformal metric. If $L_\gamma(\rho)$ is the length of the curve that separates γ from ζ , then the divergence of $\int \frac{d\rho}{L_\gamma(\rho)}$ guarantees the weakness of γ .

The criterion has the following corollary, a counterpart of Laasonen's parabolicity criterion [3]. Using the universal covering surface W^∞ of W , one uniformizes W by mapping W^∞ conformally onto the disk $|w| < 1$. Take the fundamental polygon B_0 that contains $w = 0$, bounded by an infinite number of circular arcs orthogonal to $|w| = 1$. The intersection of $|w| = r (< 1)$ and B_0 consists of a finite set of arcs, corresponding to $S(\rho)$ of the above

test if the hyperbolic metric $dp = \frac{|dw|}{1-|w|^2}$ is used. Let $\ell_\gamma(r)$ be

the Euclidean length of those arcs that separate γ from $w = 0$.

Then γ is weak if $\int \frac{dr}{\ell_\gamma(r)} = \infty$.

The above tests are valid for arbitrary Riemann surfaces.

Now suppose W is a planar covering surface of the complex z -plane

with a finite number of traces z_i of branch points. Draw a

Jordan curve C through the z_i . The counterpart of C on the

sheets of W decomposes each sheet into two "half sheets". Their

interconnections can be described by a topological tree, which

separates the z -plane into disjoint regions corresponding to the

various boundary components. Take a nested sequence of sub-

regions and count the numbers $\delta_{n\gamma}$ of knots on their boundaries.

A counterpart of Wittich's parabolicity test [17] is as follows: the

divergence of $\sum 1/\delta_{n\gamma}$ is a sufficient condition for γ to be weak.

Suppose now W is a plane region. Take a sequence of dis-

joint doubly-connected regions $E_n \subset W$ such that E_n separates

E_{n-1} from γ . Let d_n denote the (minimal) distance between the

two boundary components γ_n and γ'_n of E_n . Let L_n signify the

infimum of the (Euclidean) lengths of rectifiable curves a separ-

ating γ_n and γ'_n such that the distance from a to $\gamma_n \cup \gamma'_n$ is

$\geq d_n/2$. The "relative width" of E_n is denoted by $\beta_n = d_n/L_n$ [5].

If $\sum \beta_n = \infty$, then γ is weak.

Finally, we mention an application of the method of Ahlfors and Nevanlinna [4] that makes use of square nets. Given W and γ as above, cover the plane by a net of squares with sides 2^{-n} . The subset of closed squares that meet β decomposes into disjoint components, and we count the number k_n of squares in the component that contains γ . Then γ is weak if $\sum 1/k_n = \infty$.

9. In this fashion we have various ways of getting information on strong or weak components γ . Regarding the third problem, instability of γ , I have so far no results to report.

That the problem of the nature of a boundary component is difficult is not so surprising, when one realizes that the classical type problem of a simply-connected region is a special case of our problem; even in this simple case the solution, in spite of efforts of analysts during several decades, is not known.

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Approximation of a given function of a complex variable by simpler functions as in the Taylor development, is important both as a tool for theoretical investigations and for numerical analysis. These simpler functions may be polynomials or rational functions, or merely functions bounded in a certain region. Results established in recent years have a certain precision and also a wide range of applicability -- indeed some of these are of significance whenever a sequence of functions analytic and bounded in a region converges in a closed subregion.

I

As a first example we cite that of the Taylor development [3].

If a function $f(z)$ is analytic in a circle $|z| < \rho$ but not analytic throughout any larger concentric circle, we have (Cauchy-Hadamard)

$$(1) \quad f(z) \equiv a_0 + a_1 z + a_2 z^2 + \dots, \quad |z| < \rho,$$

$$(2) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1/\rho.$$

If we choose $1 < \rho < R$, we have

$$(3) \quad \limsup_{n \rightarrow \infty} [\max_{|z| \leq 1} |f(z) - s_n(z)|]^{1/n} = 1/\rho.$$

$$s_n(z) \equiv a_0 + a_1 z + \dots + a_n z^n.$$

That the first member of equation (3) is not greater than the second member follows from (2) as a matter of simple algebraic inequalities, and the strong inequality would contradict (2).

If we set $|a_n| \leq A/\rho_1^n$, $1 < \rho_1 < \rho$, we have

$$(4) \quad [\max_{|z| \leq R} |s_n(z)|] \leq M_n = A_1 R^n / \rho_1^n,$$

whence from (3), denoting by E the set $|z| \leq 1$,

$$(5) \quad \limsup_{n \rightarrow \infty} [\max_{z \text{ on } E} |f(z) - s_n(z)|]^{1/\log M_n} \\ \leq \exp[-\log \rho / (\log R - \log \rho_1)].$$

The functions $s_n(z)$ depend on a sequence $n = 1, 2, \dots$ rather than on a continuous parameter, but if we introduce the notation

$$f_M(z) \equiv s_n(z), \quad M_n \leq M < M_{n+1},$$

we may write

$$(6) \quad |f_M(z)| \leq M, \quad |z| < R,$$

$$\limsup_{M \rightarrow \infty} [\max_{z \text{ on } E} |f(z) - f_M(z)|]^{1/\log M} \\ \leq \exp[-\log \rho / (\log R - \log \rho_1)].$$

This last inequality persists if we change notation so that $f_M(z)$ now indicates a function (which exists by the theory of normal families) analytic in $|z| < R$ and satisfying (6), of best approximation to $f(z)$ on E . Then we have $(\rho_1 \rightarrow \rho)$

$$(7) \quad \limsup_{M \rightarrow \infty} [\max |f(z) - f_M(z)|, z \text{ on } E]^{1/\log M} \leq \exp[-\log \rho / (\log R - \log \rho)]$$

We have established a part of

THEOREM 1. If $f(z)$ is analytic in $|z| < \rho$ but not throughout the interior of any $|z| < \rho'$, $(\rho' > \rho)$, and if we have $1 < \rho < R$, then there exists a set of functions $f_M(z)$ analytic in $|z| < R$ satisfying (6) and

$$(8) \quad \limsup_{M \rightarrow \infty} m_M^{(1/\log M)} = \exp[-\log \rho / (\log R - \log \rho)],$$

$$m_M = \max[|f(z) - f_M(z)|, z \text{ on } E].$$

Conversely, if there exists a family of functions $f_M(z)$ analytic in $|z| < R$ satisfying (6) and (8), then $f(z)$ is analytic throughout $|z| < \rho$.

Inequality (7) shows that the first member of (8) is not greater than the second member. A strong inequality here would contradict the second part of the theorem, to the proof of which we now turn. If we choose the values $M = e^n$, $n = 1, 2, \dots$, and set $f_{M_n}(z) \equiv \phi_n(z)$,

equation (8) implies

$$(9) \quad \limsup_{n \rightarrow \infty} [\max |f(z) - \phi_n(z)|, z \text{ on } E]^{1/n} \\ \leq \exp[-\log \rho / (\log R - \log \rho)],$$

$$(10) \quad |\phi_n(z)| \leq e^n, \quad |z| < R.$$

Here we may write

$$(11) \quad \limsup_{n \rightarrow \infty} [\max |\phi_n(z) - \phi_{n-1}(z)|, |z| = 1]^{1/n} \\ \leq \exp[-\log \rho / (\log R - \log \rho)],$$

$$(12) \quad \limsup_{n \rightarrow \infty} [\max |\phi_n(z) - \phi_{n-1}(z)|, |z| = R]^{1/n} \leq e.$$

The Hadamard three-circle theorem expresses the fact that for a function $F(z)$ analytic in an annulus, $\log[\max |F(z)|, |z| = r]$ is a convex function of $\log r$; this conclusion applies also to $\limsup [\log \max |\phi_n(z)|, |z| = r]$ of a sequence of such functions. From (9) and (10) there follows by (11) and (12)

$$(13) \quad \limsup_{n \rightarrow \infty} [\max |\phi_n(z) - \phi_{n-1}(z)|, |z| = \rho]^{1/n} \leq 1,$$

and if the strong inequality holds in (11) or (12) it also holds in (13), so the sequence $\phi_n(z)$ converges uniformly throughout some

$|z| < \rho$, $\rho' > \rho$. In any case if we replace $|z| = \rho$ in (13) by $|z| = \rho - \varepsilon < \rho$, the first member is less than unity, so the sequence $\phi_n(z)$ converges uniformly, and by (8) the functions $f_M(z)$ approach $f(z)$ uniformly on E , so $f(z)$ is analytic in $|z| < \rho$. This completes the proof of Theorem 1.

II

The question presents itself as to whether Theorem 1 extends to approximation on a more general point set E , by functions required to be analytic and bounded in a more general region D containing E . Here we have [1]

THEOREM 2. Let D be a bounded region whose boundary C consists of a finite number of mutually disjoint Jordan curves, and let E with boundary B be a set consisting of a finite number of closed mutually disjoint Jordan regions interior to D . Let $u(z)$ be the function harmonic in $D - E$, continuous in the closure of $D - E$, equal to zero and unity on B and C respectively. Denote by D_σ generically the set $0 < u(z) < \sigma$ in D .

Let $f(z)$ be analytic in $E + D_{\rho'}$, but not throughout any $E + D_{\rho'}$, $\rho' > \rho$. There exists a set of functions $f_M(z)$ defined for every $M (> 0)$ analytic and in modulus not greater than M in D , such that

$$(14) \quad \begin{aligned} m_M &= \max[|f(z) - f_M(z)|, z \text{ on } E], \\ \limsup_{M \rightarrow \infty} m_M^{(1/\log M)} &= e^{-\rho/(1-\rho)}. \end{aligned}$$

Conversely, if there exists a set of functions $f_M(z)$ analytic and in modulus not greater than M in D , such that (14) is valid, then $f(z)$ is analytic throughout $E + D_{\rho}$.

To prove Theorem 2 we use a sequence of rational functions whose poles are equally distributed over a level locus of $u(z)$ near but exterior to D (we may assume C composed of analytic Jordan curves, so that $u(z)$ can be extended harmonically across C) and which interpolate to $f(z)$ in points equally distributed over B . Here equal distribution on the level loci of $u(z)$ is with respect to the harmonic function conjugate to $u(z)$. These rational functions play the role of the $s_n(z)$ in Theorem 1, and it can be shown that the first member of (14) is not greater than the second member. The equality in (14) and the second part of Theorem 2 are proved by use

of the two-constant theorem, in precisely the manner in which the three-circle theorem was used in proving Theorem 1.

From Theorem 2 it follows that if $F_M(z)$ is any family of functions analytic and in modulus not greater than M in D , we have

$$(15) \quad \limsup_{M \rightarrow \infty} [\max |f(z) - F_M(z)|, z \text{ on } E]^{1/\log M} \geq e^{-\rho/(1-\rho)}.$$

This result is obviously of great generality and applies under very weak hypotheses. Analogues of this result apply to Theorems 3, 4, and 5 below.

III

Another kind of theorem (Problem a) applies when the function approximated on E is not analytic on E but has certain continuity properties there. We say that a function $f(z)$ analytic interior to an analytic Jordan curve B and continuous in the closure E of B is of class $L(p, \alpha)$ on E , where β is a non-negative integer and $0 < \alpha < 1$, provided $f(z)$ has a β th derivative with respect to arc length on B , which satisfies there a Lipschitz condition of order α . Here the fundamental result [4] is (the numbers A_j are constants)

THEOREM 3. Let E be the closed interior of an analytic Jordan curve, and let the bounded region D contain E . If $f(z)$ is of class $L(p, \alpha)$ on E , then there exist functions $f_n(z)$ analytic in D satisfying

$$(16) \quad |f_n(z)| \leq A_1 R^n, \quad z \text{ in } D,$$

$$(17) \quad |f(z) - f_n(z)| \leq A_2 / n^{p+\alpha}, \quad z \text{ on } E.$$

Conversely, if there exist functions $f_n(z)$ analytic in D satisfying (16) and (17), then $f(z)$ is of class $L(p, \alpha)$ on E .

If the function $f(z)$ is given of class $L(p, \alpha)$, it can be shown that the $f_n(z)$ may be chosen as polynomials in z of respective degrees n , and are thus independent of D . If the $f_n(z)$ satisfying (17) are given as polynomials of respective degrees n in z , inequality (16) follows automatically by the generalized Bernstein lemma for polynomials. We omit the details of the proof.

Theorem 3 extends [5] to the case where $f(z)$ is defined merely on an analytic Jordan curve C , and D may be taken as an annular region containing C . The existence of the $f_M(z)$ analytic

in D and satisfying (16) and (17) is necessary and sufficient that $f^{(p)}(z)$ exist on C and satisfy there a Lipschitz condition of order α .

IV

Problem β partakes of the nature of both Problem A (of which Theorems 1 and 2 are examples) and of Problem α , exemplified by Theorem 3. Problem β is well illustrated by

THEOREM 4. Let $f(z)$ be analytic in $|z| < \rho$ ($\rho > 1$) and of class $L(p, \alpha)$ on C : $|z| = \rho$. Let D be the region $|z| < R$ ($R > \rho$). Then there exist functions $f_n(z)$ analytic in D satisfying

$$(18) \quad |f_n(z)| \leq AR^n / \rho^n n^{p+\alpha}, \quad z \text{ in } D,$$

$$(19) \quad |f(z) - f_n(z)| \leq A_1 / \rho^n n^{p+\alpha}, \quad z \text{ on } E: |z| = 1.$$

The geometric ratios of Theorems 1 and 2 are here refined, thanks to the smoothness properties of $f(z)$ on C .

The usual equation for the remainder of the Taylor series after $n+1$ terms can be written

$$f(z) - s_n(z) \equiv \frac{1}{2\pi i} \int_C \frac{z^{n+1} f(t) dt}{t^{n+1} (t-z)}, \quad |z| < \rho.$$

In particular for an arbitrary polynomial $q_n(z)$ of degree n (for which $q_n(z)$ itself is the sum of the first $n+1$ terms of the Taylor development) we have

$$0 \equiv \frac{1}{2\pi i} \int_C \frac{z^{n+1} q_n(t) dt}{t^{n+1} (t-z)}, \quad |z| < \rho.$$

Addition of these two equations yields

$$(20) \quad f(z) - s_n(z) \equiv \frac{1}{2\pi i} \int_C \frac{z^{n+1} [f(t) - q_n(t)] dt}{t^{n+1} (t-z)}, \quad |z| < \rho.$$

By a result from which Theorem 3 may be proved, a result concerning approximation by trigonometric polynomials in the real domain, it follows that $q_n(z)$ exist so that we have

$$(21) \quad |f(t) - q_n(t)| \leq A_2 / n^{p+\alpha}, \quad z \text{ on } C,$$

so (20) yields at once for z on E

$$(22) \quad |f(z) - s_n(z)| \leq A_2 / \rho^n n^{p+\alpha}.$$

In addition to (20) we may write

$$f(z) - s_{n+1}(z) \equiv \frac{1}{2\pi i} \int_C \frac{z^{n+2} [f(t) - q_n(t)] dt}{t^{n+2} (t-z)}, \quad |z| < \rho,$$

whence by (20)

$$s_{n+1}(z) - s_n(z) \equiv a_{n+1} z^{n+1} \equiv \frac{1}{2\pi i} \int_C \frac{z^{n+1} [f(t) - q_n(t)] dt}{t^{n+2}},$$

an equation which by virtue of (21) shows

$$(23) \quad |a_{n+1}| \leq A_3 / \rho^{n+1} (n+1)^{p+\alpha}.$$

For $|z| \leq R$ we have by (23)

$$|s_n(z)| \leq \sum_0^n |a_k| R^k \leq A_3 \sum_0^n (R^k / \rho^k) k^{p+\alpha},$$

whence (18) follows with $f_n(z) = s_n(z)$, and (19) follows from (22).

The conditions involved in Theorem 4 are relatively delicate, and the corresponding study in the geometric situation of Theorem 2 is not easy. One is inclined to use the method of Theorem 2, that of rational functions interpolating to the given function in points equally distributed on B , having poles in points equally distributed on a level locus of $u(z)$ near C . Such rational functions exist, and the analogue of (19) follows readily but not the analogue of (18). Unlike the situation for polynomials, here the difference between two approximating rational functions $r_{n+1}(z)$ and $r_n(z)$ of respective degrees $n+1$ and n is a rational function

of degree $2n+1$, and (18) does not follow. If we use not equidistribution of points on B and C but uniform distribution (say the distribution of points $e^{in\phi}$ on the unit circumference, where $\phi/2\pi$ is irrational), then $r_{n+1}(z) - r_n(z)$ remains of degree $n+1$ but the asymptotic properties of the uniformly distributed points seem not to yield (19), and indeed it seems that no distribution of points on B and C may exist yielding (18) and (19) simultaneously.

This difficulty can, however, be avoided, by (i) introducing [8] a new canonical map for multiply connected regions, and (ii) designing [7] a new series of interpolation of rational functions pertaining to the new canonical region, rational functions whose zeros (points of interpolation) are distributed among only a finite number of points and whose poles are likewise so distributed. This new series, a broad generalization of Taylor's series, enables us to treat the delicate properties required. The new theorem [9] is

THEOREM 5. Let D be a bounded region of the z -plane whose boundary consists of mutually disjoint analytic Jordan curves $B_1, B_2, \dots, B_\mu, C_1, C_2, \dots, C_\nu$, and let $\bar{U}(z)$ be the function harmonic in D , continuous in the closure of D , and equal to zero and unity on $B = \sum B_j$

and $C = \Sigma C_j$ respectively. For every σ , $0 < \sigma < 1$, let Γ_σ denote the locus $\overline{U}(z) = \sigma$ in D , and let D_σ denote the subregion $0 < \overline{U}(z) < \sigma$ of D , whose boundary is $B + \Gamma_\sigma$.

If Γ has no multiple points, and if the function $f(z)$ is analytic on D plus the closed interiors of the B_j , and is of class $L(p, \alpha)$, $0 < \alpha < 1$, on Γ_p , then there exist functions $f_n(z)$ analytic in D plus the closed interiors of the B_j such that

$$(24) \quad |f(z) - f_n(z)| \leq A_1 e^{(-n\rho/\tau)} n^{p+\alpha}, \quad z \text{ on } B,$$

$$(25) \quad |f_n(z)| \leq A_2 e^{[n(1-\rho)/\tau]} n^{p+\alpha}, \quad z \text{ in } D,$$

where $2\pi\tau$ is the total variation along Γ_p of the function conjugate to $U(z)$.

Reciprocally, if $f(z)$ is defined on B , if the functions $f_n(z)$ are analytic in D plus the closed interiors of the B_j , and if (24) and (25) are valid for some integer p (> 0) and $0 < \alpha < 1$, where τ is arbitrary, then $f(z)$ can be defined so as to be analytic in D plus the closed interiors of the B_j , and of class $L(p-1, \alpha)$ on Γ_p .

A class $L(p, \alpha)$ can be defined also for every negative integer p and $0 < \alpha < 1$, namely we say that $f(z)$ is of class $L(p, \alpha)$ on D provided $f(z)$ is analytic in a one-sided neighborhood of D_p interior to D_p within which we have $|f(z)| < A_0(p-\sigma)^{-p-\alpha}$ for z on Γ_σ , $\sigma < p$. With this definition, the two parts of Theorem 5 remain valid with no restriction on the integer p , and $0 < \alpha < 1$.

The loss of one unit in p as we change from the direct theorem to the indirect is inherent in the nature of the subject, as examples show.

For detailed proofs and for further refinements and references to the literature, the reader may refer to the most recent paper [9] on the subject.

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Seminar IV. THEORY OF AUTOMORPHIC FUNCTIONS

INDUCED REPRESENTATIONS

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This is a brief summary of the results contained in [3]. We are dealing with a complex analogue of the following situation in the realm of finite groups: If U is a subgroup of a finite group G , there is an inducing function I , which converts U -modules into G -modules. By definition I assigns to the U -module E , a G -module $I \cdot E$, consisting of the function $f: G \rightarrow E$ subject to the identity:

$$1. \quad f(gu) = u^{-1} \cdot f(g) \quad g \in G, u \in U.$$

(The group G acts on $I \cdot E$, via left translations: $(g \cdot f)x = f(g^{-1}x)$).

The module $I \cdot E$ has a certain duality property with respect to E , which is expressed by the Frobenius identity:

$$2. \quad \text{Hom}_G(W, I \cdot E) = \text{Hom}_U(W, E),$$

valid for a G -module W , and a U -module E .

This entire construction, as well as the formula 2, have the following extended analogue in the realm of complex analytic groups.

Assume that U and G are complex-analytic Lie-groups, and that U is closed in G . Also let E be a holomorphic U -module. Let $G \times_U E$ denote the vector-bundle over G/U defined by E , and finally let $H^*(G/U; SE)$ be the cohomology module of G/U with

coefficients in the sheaf of local analytic sections of $Gx_U E$. Then $H^*(G/U; SE)$ is in a natural way a G -module (induced by the left translations of G on $Gx_U E$.) The correspondence $E \rightarrow H^*(G/U; SE)$ is the desired analogue of I. Indeed $H^0(G/U; SE)$ can alternately be described as the module of holomorphic maps $f: G \rightarrow E$, subject to 1. Suppose now that:

- a. G is semi-simple, $\pi_0(G) = \pi_1(G/U) = 0$
- b. G/U is compact.

THEOREM I. Under these conditions

$$3. \quad \text{Hom}_G(W, H^*(G/U, SE)) = H^*(u, u \cap \bar{u}, \text{Hom}(W, E)).$$

(Here W is a holomorphic G -module, while u denotes the Lie-algebra of U . The bar denotes complex conjugation with respect to a maximal compact subgroup of G .)

In dimension zero, 3. gives a precise analogue of I:

$$4. \quad \text{Hom}_G(W, H^0(G/U, SE)) = \text{Hom}_U(W, E).$$

and this formula in turn yields a theorem of Borel-Weil rather directly [2].

INDUCED REPRESENTATIONS

Other consequences of 3. are:

5. The Euler number $\chi(G/U; SE)$ depends only on the C^∞ structure of $G \times_U E$. In particular if the usual Euler number for G/U vanishes, then also $\chi(G/U; SE) = 0$ for any holomorphic U -module E . (Here $\chi(G/U; SE) = \sum (-1)^i \dim H^i(G/U; SE)$.)

When $\chi(G/U) \neq 0$, the manifold G/U is known to be a projective algebraic variety. In that case $\chi(G/U; SE)$ has been computed by Borel-Hirzebruch [1]. Their answer led them to a conjecture which is confirmed by the following theorem:

THEOREM II. Let G/U be algebraic, Then if E is an irreducible U -module, $H^*(G/U; SE)$ is either an irreducible G -module, or the zero-module.

(For a detailed formula in terms of the maximal weights of E see [3].)

This theorem in dimension 0 (i.e. for $H^0(G/U; SE)$) is precisely the result of Borel-Weil alluded to earlier.

It is not difficult to see that Theorems I and II together imply that $H^*(G/U; SE)$ is finitely computable in general, even in the case

when G/U is not algebraic.

As an example of an explicit computation we may cite the following:

THEOREM III. Let X be a compact homogeneous Kaehler manifold, and let θ be the analytic tangent-bundle of X . Then $H^i(X; S\theta) = 0$ for $i \geq 1$.

In view of the results of Froelicher-Nijenhuis, the vanishing of $H^1(X; S\theta)$ then implies that the analytic structure of all such manifolds is locally rigid.

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The bounded homogeneous symmetric domains were classified by E. Cartan [2]. With each domain X of this type one has to consider a certain algebraic manifold X' which is a compact homogeneous hermitian symmetric space. X is imbeddable in X' as open subset such that each automorphism of X can be extended to an automorphism of X' . The paper [1] gives methods for the calculation of the Chern numbers of X' and in [4] it was proved that X and X' are proportional with respect to Chern numbers. Here I shall review some of the results of [1] and [4] and give some applications, in particular an explicit formula for the Euler number of a symplectic manifold in the sense of Siegel [5]. It was stated in Siegel [5] at the end of section 18 that such an explicit formula was still missing.

I refer to [1] and [4] for details and more references to the literature.

1. Let X be a complex manifold which is complex-analytically homeomorphic to a bounded homogeneous symmetric domain in C^m and let

$$K(z, \bar{z}) dz_1 \overline{dz_1} dz_2 \overline{dz_2} \dots dz_m \overline{dz_m} \quad (\bar{K} = (-1)^m K)$$

be an invariant real volume element of X , written in local coordinates. There exists one (Bergman's kernel function). "Invariant" always means invariant with respect to the group of all automorphisms of X . All products of differential forms are here and always taken in the sense of exterior calculus, except in the case of a metric form ds^2 .

The Bergman metric of X is given by

$$ds^2 = \sum_{\alpha, \beta} \frac{\partial^2 \log K}{\partial z_\alpha \partial \bar{z}_\beta} (dz_\alpha \overline{dz}_\beta).$$

ds^2 is independent of the invariant volume element; it actually depends only on X as homogeneous complex manifold and has nothing to do with an imbedding of X in C^m .

The Bergman metric is an invariant kählerian metric and its curvature tensor is an invariant form of type $(1,1)$ on X with coefficients in the complex vector bundle $T \otimes T^*$ where T is the tangential complex vector bundle of X . Locally the curvature tensor can be represented as a matrix (Ω_{rs}) of $(1,1)$ -forms. We introduce on X the (total) Chern form

$$c = 1 + c_1 + \dots + c_m = \det(\delta_{rs} - \frac{1}{2\pi i} \Omega_{rs}).$$

c is an invariant closed real form of mixed degree on X and

c_k is an invariant closed real form of type (k, k) on X .

To any Kählerian metric

$$ds^2 = \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} (dz_{\alpha} d\bar{z}_{\beta}),$$

there is associated the closed real $(1, 1)$ -form

$$\omega = \frac{i}{2} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz_{\alpha} d\bar{z}_{\beta},$$

and $\frac{\omega^m}{m!}$ is the real volume element belonging to the Riemannian metric ds^2 .

For the Bergman metric on X we have

$$(1) \quad c_1 = -\frac{\omega}{\pi}.$$

For any partition $(\lambda) = (\lambda_1, \dots, \lambda_r)$ of m the form

$c_{(\lambda)} = c_{\lambda_1} c_{\lambda_2} \dots c_{\lambda_r}$ is an invariant form of type (m, m) on X and thus a real multiple of the (invariant) volume element $\frac{\omega^m}{m!}$ of the

Bergman metric. We write

$$(2) \quad c_{(\lambda)} = (-\pi)^{-m} d_{(\lambda)} \frac{\omega^m}{m!}.$$

If Δ is a discontinuous group of automorphisms of X which

has no fixed points and for which X/Δ is compact, then X/Δ is a non-singular algebraic manifold whose characteristic numbers of Chern are given by the formula

$$c_{(\lambda)}[X/\Delta] = (-\pi)^{-m} d_{(\lambda)} \cdot V(X/\Delta)$$

where $V(X/\Delta)$ is the volume of X/Δ with respect to the Bergman metric. In particular, we have for the Euler number of X/Δ

$$E(X/\Delta) = (-\pi)^{-m} d_m \cdot V(X/\Delta).$$

We have written here d_m instead of $d_{(m)}$.

It is our intention to give a method for the calculation of the numbers $d_{(\lambda)}$. This was essentially done in [4]. But we will give here some more explicit results, particularly for the number d_m .

2. From now on we assume that X is equivalent to an irreducible bounded homogeneous symmetric domain. Then X can be considered in a canonical fashion as open subset of a homogeneous algebraic variety $X' = G/U$ belonging to one of the following types:

- I. $U(p+q) / U(p) \times U(q)$
- II. $SO(2n)/U(n)$

- III. $Sp(n)/U(n)$
- IV. $SO(n+2) / SO(n) \times SO(2)$
- V. $E_6 / Spin(10) \times T^1$
- VI. $E_7 / E_6 \times T^1$

By the proportionality theorem of [4] we get using (1)

$$(3) \quad d_{(\lambda)} = m! \, c_{(\lambda)}[X'] / c_1^m[X'].$$

We see that $d_{(\lambda)}$ is a rational number. For the calculation of $d_{(\lambda)}$ one has to calculate the Chern numbers $c_{(\lambda)}[X']$ and $c_1^m[X']$ of X' . To avoid confusion note that in the expression $c_{(\lambda)}[X'] = c_{\lambda_1} c_{\lambda_2} \dots c_{\lambda_r}[X']$ the symbol c_j denotes, of course, the j^{th} Chern class of X' which is an element of $H^{2j}(X', \mathbb{Z})$.

In particular we have

$$(4) \quad d_m = m! \, E(X') / c_1^m[X'].$$

The Euler number of X' is well known. We give in sections 3 and 4 a method for the calculation of $c_1^m[X']$.

3. Let $Y = G/U$ be an algebraic manifold of one of the types I. - VI. and let m be the complex dimension of Y . The space Y is

simply connected, the group $H^2(Y, \mathbb{Z})$ is infinite cyclic. Let g be that generator of $H^2(Y, \mathbb{Z})$ which is positive in the sense of Kodaira. Let D be a divisor of Y with characteristic class g . The divisor D is uniquely determined up to linear equivalence. The complete linear system $|rD|$ induces for $r \geq 1$ an imbedding of Y in the projective space $P_{N_r}(C)$ where

$$N_r = \dim |rD|.$$

The degree v_r of this imbedding is given by

$$v_r = r^m \cdot v \quad \text{with } v = g^m[Y].$$

We have $v_1 = v$ and we put $N_1 = N$, and v is the degree of the imbedding of Y in $P_N(C)$ induced by $|D|$.

As well known

$$(5) \quad \lim_{r \rightarrow \infty} \frac{N_r + 1}{r^m} = \frac{v}{m!}.$$

The theorem of Riemann-Roch shows [1] that $N_r + 1$ equals the degree of a certain irreducible representation ψ_r of G . The highest weight of ψ_r equals $r\tilde{g}$ where \tilde{g} is the highest weight of ψ_1 .

Take all positive roots of G (with respect to some lexicographic order) and let b_1, \dots, b_m be those positive roots of G

which do not belong to U . Assume that the notations are such that

b_1 is the (unique) positive simple root of G among the b_j . Then

\tilde{g} is the weight characterised by

$$2(\tilde{g}, b_1) = (b_1, b_1),$$

$$(\tilde{g}, \phi) = 0 \text{ for all positive simple roots } \phi \text{ of } G \text{ different from } b_1.$$

Let a be the sum of all positive roots of G and put

$$b = \sum_{j=1}^m b_j.$$

The following formula can be obtained by using known formulas of representation theory (compare [1], [4]) and observing that, when expressing b_j as linear combination of positive simple roots, the coefficient of b_1 equals 1.

$$N_r + 1 = \prod_{k=1}^m \frac{(a, b_k) + r(b_1, b_1)}{(a, b_k)}.$$

We obtain by (5)

$$(6) \quad v = g^m[Y] = \frac{m! (b_1, b_1)^m}{\prod_{k=1}^m (a, b_k)}.$$

The first Chern class of Y is given by the weight b , see [1]. This yields, since b is an integral multiple of g ,

$$(7) \quad c_1(Y) = \frac{2(b, b_1)}{(b_1, b_1)} \cdot g.$$

Thus we have using (6)

$$(8) \quad c_1^m[Y] = \frac{m! 2^m (b, b_1)^m}{\prod_{k=1}^m (a, b_k)}.$$

The formula (6) can be put in an especially convenient form if G is equally laced, i.e. all simple roots of G have equal length. For a positive root w let $\mu(w)$ be the sum of coefficients of w in its expression as linear combination of positive simple roots. Taking into account that $(a, \phi) = (\phi, \phi)$ for a positive simple root ϕ , we get

$$(9) \quad v = m! \prod_{k=1}^m \mu(b_k)$$

and also

$$(10) \quad N_1 + 1 = N + 1 = \prod_{k=1}^m \frac{\mu(b_k) + 1}{\mu(b_k)}.$$

4. As examples let us consider the type III and the exceptional types V and VI.

$$\text{III. } Y = \text{Sp}(n)/U(n)$$

$$\dim Y = m = \frac{n(n+1)}{2},$$

$$v = 2^m m! / \prod_{1 \leq i < j \leq n} (i+j) = 2^m m! \prod_{k=1}^n \frac{k!}{(2k)!}$$

$$c_1(Y) = (n+1)g, \quad E(Y) = 2^n$$

$$\text{V. } Y = E_6 / \text{Spin}(10) \times T^1$$

$$\dim Y = m = 16$$

E_6 is equally laced. The 16 integers $\mu(b_j)$ are

$$1, 2, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 10, 11$$

Thus by (9) and (10)

$$v = 78, \quad N = 26$$

$Y = E_6 / \text{Spin}(10) \times T^1$ is an algebraic variety of degree 78 in $P_{26}(C)$.

Its Euler number is 27. By (7) one can obtain $c_1(Y) = 12g$.

$$\text{VI. } Y = E_7 / E_6 \times T^1$$

$$\dim Y = m = 27$$

E_7 is equally laced. The 27 integers $\mu(b_j)$ are

1, 2, 3, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 14, 15, 16, 17.

Thus by (9) and (10)

$$v = 13110, \quad N = 55.$$

$Y = E_7 / E_6 \times T^1$ is an algebraic variety of degree 13110 in $P_{55}(C)$.

Its Euler number is 56. By (7) one can obtain that $c_1(Y) = 18g$.

The values for v and N in the cases V, VI are mentioned in

E. Cartan [3] p. 160 and p. 1255 respectively.

5. We return now to the calculation of the numbers $d_{(2)}$ of a complex manifold X equivalent to a bounded homogeneous symmetric domain. This problem is in principle solved by formula (3), since the Chern numbers of X' can be calculated by methods of [1]. The explicit formulas of sections 3 and 4 and the known values of the Euler number of X' enable us by formula (4) to calculate the number d_m of X . Here we carry through the calculations only for the case where X' belongs to type III, i. e. X is equivalent to Siegel's generalized upper half-plane [5]. The dimension of X and X' equals $m = \frac{n(n+1)}{2}$. We obtain by formula (4) and section 4

$$(11) \quad d_m = 2^{\frac{1}{2}n(n-1)} \cdot (n+1)^{-m} \prod_{k=1}^n \frac{(2k)!}{k!} \quad (\text{type III}).$$

Siegel defines his upper half-plane X as the space of all symmetric $n \times n$ complex matrices z with the property that the imaginary part of z is positive definite. Put

$$z = x + iy$$

and let x_{rs} be the coefficients of x and \tilde{y}_{rs} those of y^{-1} . Then Siegel introduces the invariant volume element

$$(12) \quad v_S = \pm \prod_{1 \leq r < s \leq n} (dx_{rs} dy_{rs}),$$

the sign chosen in such a way that v_S is a positive multiple of the Bergman volume element $\frac{\omega^m}{m!}$ of X . One can prove that

$$(13) \quad \frac{\omega^m}{m!} = 2^{\frac{1}{2}n(n-1)} \cdot \frac{(n+1)^m}{2^{2m}} \cdot v_S.$$

By (2), (11), (13) we get the following theorem:

THEOREM. Let X be Siegel's generalized upper half plane ($\dim X = m = \frac{n(n+1)}{2}$). Let v_S be Siegel's volume element (12). Then the Chern-Euler form c_m of X is given by the formula

$$c_m = (-\pi)^{-m} \cdot \gamma_n \cdot V_S, \quad \text{where}$$

$$\gamma_n = 2^{-2m} \prod_{k=1}^n \frac{(2k)!}{k!}.$$

For the Euler number of a compact "symplectic" manifold M , i. e. if M is a complex manifold whose universal covering is Siegel's upper half plane X and M is of the form X/Δ , we get

$$(14) \quad E(M) = (-\pi)^{-m} \cdot \gamma_n \cdot V_S.$$

(M has dimension $m = \frac{n(n+1)}{2}$ and V_S is the volume of M in the sense of Siegel (12)).

We have

$$\gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{3}{8}, \quad \gamma_3 = \frac{45}{2^6}.$$

The preceding theorem makes Siegel's theorem 5 of [5] more explicit. Siegel writes c_n instead of γ_n . He calculated the γ_n for $n \leq 3$ but obtained for γ_3 the value $\frac{45}{8}$. It should still be checked carefully whether or not some power of 2 crept erroneously into the preceding calculations.

The question remains whether (14) is true for open

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"symplectic" manifolds with finite volume. If so, one would get for example explicit formulae for the Euler number of $X/\Gamma_n(p)$ where X is Siegel's upper half plane and $\Gamma_n(p)$ the congruence subgroup (p prime $\neq 2$) of the modular group Γ_n . This formula for the Euler number would involve Bernoulli numbers (see [5], section 45).

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AUTOMORPHIC FORMS IN HALF-SPACES

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In this note a short survey shall be given of a theory of automorphic forms in half-spaces, where in the following a half-space will be defined as a suitable generalization of the upper half of the complex plane. One tries in this way to develop a unified theory of automorphic forms of several variables, which are of interest from the standpoint of the theory of numbers, i. e. of the modular forms of Hilbert and Siegel and of the Hermitean modular forms. The investigations are based on the concept of the domain of positivity, recently investigated by the author (cp. "Positivitätsbereiche im R^n ", American Journal of Mathematics, 79, Vol. 3, 1957). Only those discontinuous subgroups of the automorphism group of the half-spaces will be considered which have a non-compact fundamental domain. For clarification of these concepts it is recommended that one go back to the above mentioned examples.

1. Let Y be an open subset of R^n with the following Properties:

(P.1) Given two arbitrary points a and b of Y ,

$$a'b > 0.$$

(P.2) If x is not contained in Y , then there is an $a \in \overline{Y}$ for which $a'x \leq 0$.

Points of R^n are represented here by columns of n elements, a' means the transposed row of a and \overline{Y} the closure of Y . In the following any open subset of R^n with these two properties is called a domain of positivity. One easily proves that any domain of positivity is convex and maximal, i. e. there is no domain of positivity properly containing it. Further, if a and b are in Y , so is $\alpha a + \beta b$ for all positive α and β , and the intersection of \overline{Y} and $-\overline{Y}$ is the null vector. One immediately sees that the domains defined in this way are also domains of positivity in the sense of N. Bourbaki (cf. Intégration, Chap. II, §1, No. 2).

Any domain of positivity induces a semi-ordering in R^n by the definition

$$x \geq y \text{ if } x - y \in \overline{Y} \text{ (and } x > y \text{ if } x - y \in Y).$$

This semi-ordering is Archimedean in our case, i. e. for every a in Y and every x in R^n there is some $\lambda > 0$ so that $\lambda a > x$.

2. By $\Sigma(Y)$ we denote in the following the (linear) auto-

morphism group of Y , i. e. the group of real non-singular matrices W of n -th order such that $y \rightarrow Wy$ induces a one-one mapping of Y onto itself. One immediately sees that, if W belongs to $\Sigma(Y)$, so does W' .

A function defined on Y we will call a norm of Y , if it has the following two properties

(N.1) $f(y)$ is continuous and positive on Y ,

(N.2) $f(Wy) = ||W|| f(y)$ for any $W \in \Sigma(Y)$,

where $||W||$ denotes the absolute value of the determinant $|W|$.

THEOREM 1: For every domain of positivity Y there exists a (real-analytic) norm $N(y)$. The element of arc length formed by means of $N(y)$

$$ds^2 = - \sum_{k,l} dy_k dy_l \frac{\partial^2}{\partial y_k \partial y_l} \log N(y)$$

is positive definite and invariant under the mappings $y \rightarrow Wy$, where W belongs to $\Sigma(Y)$.

Furthermore, there is always an involution $y \rightarrow y^*$ of Y with the properties:

$$(y^*)^* = y, \quad N(y)N(y^*) = 1, \quad (Wy)^* = W^{-1}y^*$$

for every $W \in \Sigma(Y)$, and ds^2 is invariant even under the mapping $y \rightarrow y^*$.

3. Let us suppose now that Y is homogeneous, i. e. the effect of $\Sigma(Y)$ on Y is transitive. It is easily seen that the norm of Y is then uniquely determined except for a constant factor. $\Sigma(Y)$ can be looked upon as the group of "units" of $N(y)$ in the following sense:

THEOREM 2: A real non-singular matrix W belongs to $\Sigma(Y)$ if and only if $WY \cap Y \neq \emptyset$ and $N(Wy) = ||W||N(y)$ for all $y \in WY \cap Y$.

Because of Theorem 1, Y can be looked upon as a Riemannian manifold. It can then be proved that every homogeneous domain of positivity is complete and that to any two points of Y there exists exactly one geodesic connecting them. Furthermore it can be deduced easily from the existence of the involution $y \rightarrow y^*$, that a homogeneous domain of positivity is always a weakly-symmetric space in the sense of A. Selberg.

4. It is a very important, and in the applications of the theory, often discussed problem, to construct a simple fundamental domain in Y for a given discontinuous subgroup Γ of $\Sigma(Y)$. In the present case one succeeds by the following construction: If for $a \in Y$ we define

$$F(\Gamma, a) = \{y : y \in Y, a'y \leq a'Wy \text{ for every } W \in \Gamma\}$$

then the following theorem holds:

THEOREM 3: If $W'a \neq a$ for every $W \in \Gamma$, then $F(\Gamma, a)$ is a convex fundamental domain of Γ . Every compact subset K of Y has non-void intersections with only a finite number of images of $F(\Gamma, a)$ generated by mappings of Γ .

If F is a fundamental domain of a discontinuous subgroup of $\Sigma(Y)$ we denote by F_1 the set of points y of F satisfying the relation $N(y) \leq 1$. The groups for which F_1 has a finite Euclidean volume are of special interest.

To any lattice G of R^n , one can easily construct a discontinuous subgroup of $\Sigma(Y)$. Namely, if we denote by $\Sigma(G)$ the group

of non-singular matrices W mapping G onto itself, one easily verifies that

$$\Sigma(Y, G) = \Sigma(Y) \cap \Sigma(G)$$

is a discontinuous subgroup of $\Sigma(Y)$. A lattice shall be called an admissible one, if the following two properties hold:

(L.1) There exists a number $\gamma > 0$ such that $N(a) \geq \gamma$

for all $a \in Y \cap G^*$.

(L.2) $\Sigma(Y, G)$ has a fundamental domain F in Y for which F_1 is of finite volume.

G^* denotes the complementary lattice to G , i. e. the set of points b such that $a'b$ is an integer for any $a \in G$.

5. We now come to the definition of the concept "half-space".

For this purpose we consider subsets Z of C^n , consisting for a given homogeneous domain of positivity $Y \subset R^n$, of all points

$$z = x + iy, \quad x \in R^n, \quad y \in Y.$$

We also write

$$Z = X + iY, \quad X = R^n.$$

By $\Sigma(Z)$ we denote the group of (holomorphic) automorphisms of Z . A slight generalization of S. Bergman's results shows that Z has a linear element invariant under $\Sigma(Z)$ and therefore also an invariant metric in the sense of Bergman. The corresponding invariant volume element has the form

$$dv = \frac{dx dy}{N(y)^2}$$

where $N(y)$ is the norm of Y (uniquely determined up to a constant factor).

By the special form of Z one immediately recognizes that the mappings

$$(1) \quad z \longrightarrow Wz + t, \quad W \in \Sigma(Y), \quad t \in R^n,$$

are always automorphisms of Z . But there is still one more automorphism $z \longrightarrow \omega(z)$, defining an involution of Z and such that $\omega(iy) = iy^*$. All the known examples for Z further the presumption that $\Sigma(Z)$ is generated by the mappings (1) and this involution.

In the following, a set $Z = X + iY$ shall be called a half-space, if Y is a homogeneous domain of positivity and every map-

ping $z \rightarrow a(z)$ of $\Sigma(Z)$ is of the form $a(z) = Wz + t$, provided that the functional determinant

$$\left| \frac{\partial a(z)}{\partial z} \right|$$

is constant in Z . Here, too, the possibility of deducing this last postulate is to be presumed.

6. If Γ is a subgroup of $\Sigma(Z)$, we denote by G_Γ the set of points $t \in R^n$ for which the mapping $z \rightarrow \tau(z) = z + t$ belongs to Γ . Furthermore let Ω_Γ be the set of real non-singular matrices W of n -th order for which there exists a c_W in R^n such that $\tau(z) = Wz + c_W$ belongs to Γ . Evidently G_Γ is an additive group, being discrete in R^n in the case where Γ is discontinuous. We call Γ an admissible subgroup of $\Sigma(Z)$, if the following properties hold:

- (S.1) Γ is discontinuous in Z ,
- (S.2) G_Γ is an admissible lattice in R^n ,
- (S.3) The index of Ω_Γ in $\Sigma(Y, G_\Gamma)$ is finite,
- (S.4) For any a in Z , every sequence a_v in Γ with the property

$$N(\text{Im } a_1(a)) < N(\text{Im } a_2(a)) < \dots$$

ends after a finite number of steps.

The fact that Ω_Γ is a subgroup of $\Sigma(Y, G_\Gamma)$ can be inferred from (S. 2) alone. (S. 4) means that the fundamental domain of Γ in Z is certainly not compact. Those groups which concern applications to the theory of numbers are especially important, and they certainly have not yet been sufficiently investigated from the standpoint of the theory of holomorphic functions. Postulate (S. 2) in a certain sense describes the form of the fundamental domain of Γ in the neighborhood of the ideal point. For every admissible subgroup of $\Sigma(Z)$ a simple fundamental domain exists.

A substitution $\sigma \in (Z)$ shall be called a cuspidal of Γ , if $\sigma^{-1}\Gamma\sigma$ is an admissible subgroup of $\Sigma(Z)$. In generalizing the most important discontinuous groups in several variables, we call a subgroup Γ of $\Sigma(Z)$ a modular group of Z , if there exists an $a \in Y$ and a finite number of cusps $\sigma_j \in \Gamma$ such that for every $z \in Z$ there is an α belonging to the complex

$$\bigcup_{\nu} \sigma_{\nu}^{-1} \cdot \Gamma$$

for which $\text{Im } \alpha(z) \geq a$. For every modular group the invariant

volume of a measurable fundamental domain is finite, and if Γ is a modular group each subgroup of Γ of finite index is also a modular group.

7. For every $\alpha \in \Sigma(Z)$ the functional determinant is different from zero in Z . Consequently

$$\log \left| \frac{\partial \alpha(z)}{\partial z} \right|$$

can be defined as a holomorphic function. For real r , let

$$\left| \frac{\partial \alpha(z)}{\partial z} \right|^r = \exp \left\{ r \log \left| \frac{\partial \alpha(z)}{\partial z} \right| \right\} ;$$

in the case that r is an integer the multiplication formula for functional determinants yields

$$(2) \quad \left| \frac{\partial \alpha\beta(z)}{\partial z} \right|^r = \left| \frac{\partial \alpha\beta(z)}{\partial \beta(z)} \right|^r \left| \frac{\partial \beta(z)}{\partial z} \right|^r .$$

We will confine ourselves here to the case of automorphic forms without multiplier system, and choose for that purpose a real number r for which (2) is valid without restriction. For any function $f(z)$ defined in Z and for every $\alpha \in \Sigma(Z)$ we define

$$f|_a = f(z)|_a = \left| \frac{\partial a(z)}{\partial z} \right|^r f(a(z)).$$

Because of (2) we find

$$(f|_a)|_\beta = f|(a\beta)$$

for a and $\beta \in \Sigma(Z)$.

We call a function $f(z)$ an automorphic form of dimension $-r$ for a modular group Γ of Z , if the following properties hold:

(A.1) $f(z)$ is holomorphic in Z ,

(A.2) $f|_a = f$ for every $a \in \Gamma$.

Such a form will be called an integral form, if in addition the postulate

(A.3) For every cusp σ of Γ there exists a $c_\sigma \in Y$ such that $f|_\sigma$ is bounded in the domain described by

$$\text{Im } z \geq c_\sigma,$$

is satisfied. If also the following postulate is fulfilled, $f(z)$ is called a cuspidal form:

(A.4) For every cusp σ of Γ there exists a $c_\sigma \in Y$ so that

$$\lim_{\substack{c \\ \sigma} \leq y \rightarrow \infty} \{f(z)|\sigma\} = 0.$$

These postulates are the exact generalizations of the usual ones.

The following theorems can be deduced, among others:

THEOREM 4: If σ is a cusp of a modular group Γ , then every automorphic form $f(z)$ of Γ has a Fourier series, which is absolutely and uniformly convergent in Z :

$$f(z)|\sigma = \sum_{a \in G_\sigma} A(f|\sigma; a) e^{2\pi i a' z}, \quad G_\sigma = (G_{\sigma^{-1} \cdot \Gamma \sigma})^*.$$

THEOREM 5: An automorphic form $f(z)$ of a modular group is an integral (a cusp) form if and only if for every cusp σ of Γ and for every a not belonging to \overline{Y} (to Y) the coefficients $A(f|\sigma; a)$ are zero.

THEOREM 6: If Γ is a modular group and $r < 0$, then every integral automorphic form of the dimension $-r$ is identically zero.

Furthermore, some estimations of the Fourier coefficients of cusp forms can be transferred to the general case. There

exists for each cusp form $f(z)$ a constant γ depending only on f such that

$$|A(f|\sigma;a)| \leq \gamma[N(a)]^{r/2}$$

for all cusps σ of Γ and every $a \in G_\sigma$.

The following theorem concerning the number of linearly independent cusp forms may also be of interest:

THEOREM 7: For any modular group Γ the number of linearly independent cusp forms of dimension $-r$ is finite.

8. As in the known examples, it is also possible in the general case to construct automorphic forms by means of Poincaré series. For any modular group Γ and an element $b \in G_\Gamma^* \cap \overline{Y}$ we define

$$P(\Gamma, b; z) = \sum_{a \in \Gamma}^* \left| \frac{\partial a(z)}{\partial z} \right|^r e^{2\pi i b' a(z)}$$

the sum being taken only over a which yield different terms. For the cases $b = 0$ and $b \in Y$ one shows that the series are absolutely and uniformly convergent for $r \geq 2$, and for any cusp σ of Γ ,

$$(3) \quad P(\sigma^{-1} \cdot \Gamma \cdot \sigma, b; z) | \sigma$$

is an integral automorphic form of Γ of dimension $-r$. If b is chosen in Y , the Poincaré series (3) is a cusp form. By means of a generalization of the Petersson metrization of automorphic forms, one finally can show that every cusp form of Γ can be linearly composed of Poincaré series (3).

9. Finally, it is shown how the known examples can be formulated in this theory:

HILBERT'S MODULAR GROUP: As domain of positivity Y one chooses the set of points $y \in \mathbb{R}^n$ having positive components only. Then $Z = X + iY$ is a half-space and Hilbert's modular group of a totally-real algebraic number field of dimension n is a modular group in the sense of the above definition.

SIEGEL'S MODULAR GROUP: If $A = (a_{k\ell})$ is a real symmetric matrix of order m , one puts $n = m(m+1)/2$ and associates the vector

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$$a' = (a_{11}, a_{22}, \dots, a_{mm}, a_{12}, \dots, a_{m-1, m})$$

with this matrix. If B is another such matrix one easily verifies

$$a'b = \text{trace } (AB)$$

and one immediately sees that the set of point. a for which the matrix A is positive definite is a homogeneous domain of positivity in R^n . $Z = X + iY$ then is the so-called generalized upper half-plane and the modular group of degree m is a modular group in the sense of the above definition.

In an exactly analogous way the hermitian modular group over an imaginary-quadratic number field can be looked upon as a modular group in a suitable half-space. Furthermore one recognizes that the hitherto only scarcely investigated mixed cases, e.g. the modular group of degree m over a totally-real algebraic number field, are special cases of this general theory. That fact seems to justify, in a certain sense, the richness of the conceptual scheme of the theory.

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THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS
BELONGING TO A CLASS OF REAL ZONAL
HOROCYCLIC GROUPS

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§1. R. A. Rankin [1] has recently coined the term "horocyclic group" as an English equivalent of the French "Fuchsian group of the first kind" and the German "Grenzkreisgruppe". He calls the groups "real" if all substitutions of the group preserve the real axis, and "zonal" if the group contains translations. In this paper, we shall refer to real zonal horocyclic groups as "H-groups". An H-group, then, is a group Γ of linear transformations of a complex variable such that

- (a) Γ is discontinuous in the upper half-plane but is not discontinuous at any point of the real axis.
- (b) every transformation of Γ preserves the upper half-plane.
- (c) Γ contains parabolic substitutions with fixed point ∞ .

The main object of this paper is to determine the expansions of the Fourier coefficients of automorphic forms on H-groups of a certain class by the use of the circle method. The circle method has been employed by Rademacher and Zuckerman ([2]-[5]) when

the H-group was the modular group or one of its subgroups. Here we develop the circle method for a class of H-groups defined by the following restriction: A fundamental region of the H-group shall have exactly one parabolic cusp. (This implies that the fundamental region has a finite number of sides ([7], Th. 16, p. 75).) As a consequence of this condition, there exists a number $h > 0$ such that the fundamental region with cusp at ∞ does not extend below the horizontal line at height h above the real axis.

The circle method is elementary in character, using only Cauchy's theorem and a careful choice of the path of integration. Lacking an arithmetic characterization of the parabolic points of the H-group, such as is available in the case of the modular group, we use the geometry of the fundamental region for the construction of the path of integration.

We treat entire automorphic forms of dimension r , i. e., analytic functions of a complex variable z , which are regular in the upper half-plane and satisfy there the functional equation

$$(1.1) \quad F(Vz) = \varepsilon(a, b, c, d)(-i(cz + d))^{-r} F(z), \quad c > 0$$

$$F(Sz) = e^{2\pi i a} F(z).$$

Here V and S are elements of Γ defined in §2, the multipliers ε have absolute value 1, r is real, $|\arg -i(cz + d)| < \pi$ and $0 \leq \alpha < 1$. The second equation implies a Fourier expansion for $e^{-2\pi i \alpha z} F(z)$, in which we assume there are only a finite number of terms with negative exponents:

$$(1.2) \quad e^{-2\pi i \alpha z / \lambda} F(z) = \sum_{m=-\mu}^{\infty} a_m e^{2\pi i m z / \lambda}.$$

The series converges for $\operatorname{Im} z > 0$.

Our main result is contained in the following

THEOREM. If $F(z)$ is an automorphic form of dimension $r > 0$, its Fourier coefficients a_m with $m > 0$ are given in terms of those with $m < 0$ by the formula

$$a_m = 2\pi \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{c \in C \\ c > 0}} c^{-1} A_{c, \nu}(m) \left(\frac{\nu - \alpha}{m + \alpha} \right)^{(r+1)/2} \\ \cdot I_{r+1} \left(\frac{4\pi}{C} (\nu - \alpha)^{1/2} (m + \alpha)^{1/2} \right),$$

where

$$A_{c, \nu}(m) = \sum_{d \in D_c} \varepsilon^{-1}(a, b, c, d) \exp \{ -(2\pi i / c \lambda) [(\nu - \alpha)a - (m + \alpha)d] \},$$

and

$$a_0 = 2\pi \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\substack{c \in C \\ c > 0}} c^{-1} A_{c, \nu}(0) (2\pi\nu/c)^{r+1} / (r+1)!$$

Here I_{r+1} is the Bessel function of the first kind with purely imaginary argument, and C and D_c are defined in §2.

If $r = 0$, a_m is given by a finite sum of the same type as above with $0 < c < \beta \sqrt{m}$ ($\beta = \text{const.}$) plus an error term which is bounded as $m \rightarrow \infty$.

A remark on forms of dimension $r < -2$ is presented in §5.

For a different approach, see Petersson [8].

§2. Let Γ be an H-group. The elements of Γ may be represented by unimodular matrices $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $-V = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, where a, b, c, d are real. We may identify the matrices $\pm V$ with the linear transformation $Vz = (az + b)/(cz + d)$. The subgroup of Γ consisting of all V which preserve ∞ , i. e., in which $c = 0$, is known to be a cyclic group generated by a translation $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$ ([6], 33).

Because of the discontinuity of Γ , Γ is discrete: there is no sequence of different $V_n \in \Gamma$ which tends to the identity. Let C be

the set of third coefficients in the elements of Γ , i. e.,

$$C = \{x \mid \begin{pmatrix} a & b \\ x & d \end{pmatrix} \in \Gamma, \text{ for some } a, b, d\};$$

similarly for A, B, D . Using the discreteness of Γ , Petersson ([6], 34) proved that there is no sequence of different $c_n \in C$ such that $c_n \rightarrow 0$. Using the same property, we can show that C is a discrete set.

Let D_c be the set of $d \in D$ such that Γ contains at least one matrix of the form $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ with $0 \leq -d < c\lambda$. By the argument used in the proof of the discreteness of C , it follows that the set of $d \in D$ which appear with a given c in elements of Γ is a discrete set. This means that for each $c \in C$, D_c is a finite set.

While it is not true that the remaining coefficient sets (i. e. A, B, D) are necessarily discrete, we can always transform Γ so that the transformed group $\Gamma' = L\Gamma L^{-1}$ has this property and is still an H-group. For this purpose, we select

$$L = \begin{pmatrix} \gamma^{1/2} & \gamma^{-1/2} & \delta \\ 0 & \gamma^{-1/2} & 0 \end{pmatrix}, \quad \text{where } W = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \text{ and } \gamma > 0.$$

Then $LWL^{-1} = T = \begin{pmatrix} a+\delta & -1 \\ 1 & 0 \end{pmatrix}$ belongs to the transformed group.

Now if $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then

$$TV = \begin{pmatrix} \cdot & \cdot \\ a & \cdot \end{pmatrix}, \quad T^{-1}V = \begin{pmatrix} c & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad TV^{-1} = \begin{pmatrix} \cdot & \cdot \\ d & \cdot \end{pmatrix},$$

$$V^{-1}T = \begin{pmatrix} \cdot & \cdot \\ \cdot & c \end{pmatrix}, \quad TV^{-1}T^{-1} = \begin{pmatrix} \cdot & \cdot \\ b & \cdot \end{pmatrix}, \quad T^{-1}V^{-1}T = \begin{pmatrix} \cdot & c \\ \cdot & \cdot \end{pmatrix}$$

where, e.g., $\begin{pmatrix} \cdot & \cdot \\ a & \cdot \end{pmatrix}$ denotes a matrix with a in the indicated position.

This shows that all coefficient sets are the same: $A = B = C = D$.

Since C is discrete, they all are.

We now normalize the group Γ as follows. Choose W so that $\gamma > 0$ and $|\gamma|$ is minimal. Then in the transformed group (which we now call Γ), the coefficient sets are all discrete, and if a coefficient of any element does not vanish, it has absolute value at least unity.

§3. We now introduce the restriction on Γ stated in §1.

Since Γ has exactly one equivalence class of parabolic points which, by the definition of an H-group, must contain the point ∞ , we see that all parabolic points of Γ are equivalent to ∞ . Therefore, every parabolic point is of the form $-d/c$, where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. In this representation, c and d are unique. For suppose $-d'/c'$ is the same parabolic point as $-d/c$, and let $V' = \begin{pmatrix} \cdot & \cdot \\ c' & d' \end{pmatrix}$. $V'V^{-1}$ preserves ∞ and so is equal to S^m , $V' = S^m V$, from which it follows that $c = c'$, $d = d'$.

Let $z = x + iy$ be a complex variable. We choose a closed fundamental region (FR) of Γ bounded laterally by portions of the vertical lines $x = 0, \lambda$ and bounded below by arcs of isometric circles ($|cz + d| = 1$), $c > 0$. Let R be the closed region which includes FR and all its translates by integral multiples of λ . Since $|c| \geq 1$ when $c \neq 0$, it follows that the radii of the isometric circles ($1/|c|$) do not exceed unity. Hence

$$(3.1) \quad y \geq 1 \text{ implies } z \in R.$$

Also, FR does not extend below a horizontal line of height h above the real axis (see §1), so

$$(3.2) \quad y < h \text{ implies } z \notin R.$$

We shall now describe a path which will be used later for integration. Let L_N be the line segment

$$(3.3) \quad L_N: 0 \leq x < \lambda, y = y_0 = N^{-2} h^{-1},$$

where $N > h^{-1}$ is arbitrary. Consider the sets

$$(3.4) \quad I(c, d) = \{z \in L_N \mid V_{c,d} z \in R\},$$

where $V_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. Since L_N is contained in a compact region, it is covered by a finite number of fundamental regions. Hence $I(c,d)$ is empty except for a finite set of (c,d) . Denote this set by M and then we have

$$(3.5) \quad L_N = \bigcup_{(c,d) \in M} I(c,d).$$

The sets $I(c,d)$ do not overlap except possibly at their endpoints, for no point can be mapped into the interior of R by two $V_{c,d}$ with different (c,d) .

Now let $F(z)$ be an automorphic form on Γ of the type described in §1. By Cauchy's theorem

$$(3.6) \quad a_m = \int_{L_N} e^{-2\pi i a z / \lambda} F(z) \cdot e^{-2\pi i m z / \lambda} dz = \sum_{(c,d) \in M} \int_{I(c,d)} \dots,$$

(the integrands are the same).

On each interval $I(c,d)$, we apply the transformation formula (1.1) with $V = V_{c,d}$, and in the result we introduce the Fourier series (1.2) for $F(z')$. Setting $Vz = z' = x' + iy'$, we have

$$\begin{aligned}
 a_m &= \sum_{(c,d) \in M} \varepsilon^{-1} \int_{I(c,d)} (-i(cz+d))^r \cdot \\
 &\quad \sum_{\nu=-\mu}^{-1} a_\nu e^{-2\pi i(m+a)z/\lambda + 2\pi i(\nu+a)z'/\lambda} dz \\
 (3.7) \quad &+ \sum_{(c,d) \in M} \varepsilon^{-1} \int_{I(c,d)} (-i(cz+d))^r \sum_{\nu=0}^{\infty} a_\nu e^{-2\pi i(m+a)z/\lambda + 2\pi i(\nu+a)z'/\lambda} dz \\
 &= S_1 + S_2.
 \end{aligned}$$

When $z \in I(c,d)$, we have $z' \in R$ by (3.4) and so $y' \geq h$ by (3.2); also $y = y_0$ by (3.3). Hence,

$$(3.8) \quad |cz+d|^2 = y_0/y' \leq N^{-2} h^{-2}; \quad y' \geq h.$$

With these estimates, we get

$$\begin{aligned}
 |S_2| &\leq O(N^{-r}) \sum_{\nu=0}^{\infty} |a_\nu| e^{[-2\pi|m+a|/N^2 h^2 \lambda] - 2\pi(\nu+a)h/\lambda} \\
 &\quad \sum_{(c,d)} |I(c,d)|,
 \end{aligned}$$

where $|I|$ is the length of I . Now $\sum_{(c,d)} |I(c,d)| = \lambda$. Since the infinite series converges because $h > 0$, we have

$$S_2 = O(N^{-r}).$$

In S_1 , we break up the range of (c,d) . Set $M = M_1 + M_2$, where

$$(3.9) \quad M_1 = \{(c, d) \mid 0 < c < Nh/2\}.$$

Denote the corresponding parts of S_1 by T_1 and T_2 . In T_2 we still have the estimates (3.8) and in addition

$$(3.10) \quad y' \leq y_0/c^2 y_0^2 = 1/c^2 y_0 < 4N^{-2} h^{-2} \cdot N^2 h = 4h^{-1}.$$

If M_2 is empty, we have, of course, $T_2 = 0$; otherwise

$$|T_2| \leq O(N^{-r}) \sum_{\ell=1}^{\mu} |a_{-\ell}| e^{8\pi\ell\mu/h\lambda} \sum_{(c,d) \in M_2} |I(c,d)| = O(N^{-r}).$$

Putting these results together, we get

$$(3.11) \quad a_m = \sum_{(c,d) \in M_1} \varepsilon^{-1} \int I(c,d) (-i(cz+d))^r \cdot \\ \cdot \sum_{\ell=1}^{\mu} a_{-\ell} e^{-2\pi i(m+a)z/\lambda - 2\pi i(\ell-a)z'/\lambda} dz + O(N^{-r})$$

§4. In order to make further progress, let us study the set $I(c,d)$ more closely. When $(c,d) \in M_1$, $c < Nh/2$, and $\text{Im } V_{c,d}[-d/c + iy_0] = y_0/c^2 y_0^2 > 4N^{-2} h^{-2} \cdot N^2 h = 4h^{-1} > 1$. This shows that $I(c,d)$ contains the point $-d/c + iy_0$, and by continuity, contains a closed interval $J(c,d)$ including that point. The endpoints of $J(c,d)$ are determined by considering the map of L_N by $V_{c,d}$;

this is a circle K which definitely intersects the interior of R and which leaves R for the first time at two points. The inverse images of these two points are the endpoints of $J(c, d)$. If we therefore write

$$I(c, d) = J(c, d) + J'(c, d),$$

then certainly

$$(4.1) \quad y' < 1 \text{ for } z \in J'(c, d),$$

for K is definitely below the line $y = 1$ when $z \in J'(c, d)$. Also,

(3.8) holds since $z \in I(c, d)$. The sets $\{J(c, d), J'(c, d)\}$,

$(c, d) \in M_1$, are obviously non-overlapping.

We break the sum in (3.11) into two parts, calling U_1 the sum in which the integral is extended over $J(c, d)$ and U_2 the sum in which the path of integration is $J'(c, d)$. For U_2 , we have the estimate $O(N^{-r})$, obtained in the same way as the estimate for T_2 , since (4.1) is essentially the same as (3.10) and (3.8) holds in both cases. This gives

$$(4.2) \quad a_m = \sum_{(c, d) \in M_1} \varepsilon^{-1} \int_{J(c, d)} (-i(cz + d))^r \cdot \\ \cdot \sum_{\ell=1}^{\mu} a_{-\ell} e^{-2\pi i(m+a)z/\lambda - 2\pi i(\ell-a)z'/\lambda} dz + O(N^{-r}).$$

The argument preceding (4.1) has shown incidentally that M_1 consists of all pairs (c, d) in the range $0 < c < Nh/2$.

The integrals in (4.2) must be evaluated in closed form to get our final result. For this purpose we need precise inequalities on the length of $J(c, d)$. On $J(c, d)$

$$z = -d/c + \xi + iy_0, \quad -\theta'_{c,d} \leq \xi \leq \theta''_{c,d}.$$

The point $V[-d/c + \theta''_{c,d} + iy_0]$ has an imaginary part lying between h and l . This observation leads at once to bounds for θ' , θ'' , namely,

$$(4.3) \quad 2^{-1} h^{-1/2} c^{-1} N^{-1} < \theta'_{c,d}, \quad \theta''_{c,d} \leq h^{-1} c^{-1} N^{-1}.$$

Now these bounds are substantially the same as those for the usual Farey segments, on the basis of which Rademacher and Zuckerman ([2], 439-441) evaluated the integrals analogous to those in (4.2). We can use their evaluation in our case also, and so obtain the result of our theorem when $r > 0$ and $m+a > 0$. If $m+a = 0$ (i. e., $m = a = 0$), we do not follow the approach of [2] but instead set

$$a_0 = \lim_{m \rightarrow 0, a \rightarrow 0} a_{m+a}.$$

The limit process can be carried out in all the foregoing steps and leads to the announced result.

When $r = 0$, we need to retain m in our error terms which then become $O(\exp cmN^{-2})$ with a certain constant c . The best choice is then $N = \text{const. } \sqrt{m}$, which gives a bounded error as $m \rightarrow \infty$.

§5. In the preceding discussion the positivity (or at least non-negativity) of r was essential. However, the preceding results can, in certain cases, be used to yield results for forms of negative dimension.

Let $F(z)$ now be a form of dimension $r < -2$ whose transformation formula and Fourier series are again given by (1.1) and (1.2). We construct the function

$$G(z) = \sum_{m=-\mu}^{\infty} a_m e^{2\pi i(m+\alpha)z/\lambda}, \quad \mu > 0, \quad \text{Im } z > 0,$$

where the a_m (for $m \geq 0$) are given by our theorem, and rearrange the series by the Lipschitz formula. The result is a "generalized Poincaré series" which can be seen directly to satisfy the transformation formula (1.1). Thus F and G are automorphic forms on Γ with the same principal parts and so can differ only by a

cuspidal form:

$$F(z) = G(z) + H(z),$$

where H vanishes at ∞ .

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ASYMPTOTIC FORMULAE FOR THE FOURIER COEFFICIENTS
OF MULTIPLICATIVE AUTOMORPHIC FUNCTIONS*

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1. Let $\overline{\Gamma}$ be a horocyclic group which contains parabolic substitutions and for which the upper half-plane of the complex variable $\tau = x + iy$ (x real, $y > 0$) forms a maximal domain of discontinuity. In earlier papers of the author (see [1], [2]) of the bibliography at the end of this paper), a summation method was supplied for Poincaré's series belonging to the group $\overline{\Gamma}$, the dimension -2 and to any system of multipliers v_2 of modulus 1. This method yields explicit linear representations of all automorphic forms of the same type. It makes use of the fact that Poincaré's series of dimension $-r < -2$ are uniformly absolutely convergent on each closed vertical half-strip contained in $y > 0$ and therefore define, if formed by use of suitable multipliers v_r , continuous functions both of τ and r for $y > 0$ and $2 < r \leq r_0$. The result of the method consists in the statement that these series tend, as $r \rightarrow 2 + 0$, to limiting functions which behave as if they were Poincaré series of dimension -2 converging as in the case of dimension $-r < -2$.

Considering this result, one may ask whether the method is

able to contribute to the solution of concrete problems. We investigate multiplicative automorphic forms, belonging, as above, to $\bar{\Gamma}$, $-2, v_2$ which are analytic in $y > 0$ but not integral forms so that each of them has a negative order in at least one cusp of the fundamental domain. It is shown here that the Fourier coefficients of these forms satisfy asymptotic formulae of a relatively high exactness. The relation to the Fourier coefficients of multiplicative automorphic functions (belonging to $\bar{\Gamma}, 0, v_2$) is obtained by differentiating the latter with respect to τ . The formulae may be considered as generalizations of the Hardy-Ramanujan-Rademacher results on the classical partition function $p(n)$. As Hecke's summation method is subject to rather strong restrictions, when applied to multiplicative modular functions, one has to use these new results in order to derive asymptotic formulae of the well known type on partition functions of a certain general character.

2. Let Γ be the group of real matrices $L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of determinant 1 for which $\tau \rightarrow L\tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}$ belongs to $\bar{\Gamma}$. We assume $\tau = \infty$ to be a cusp of Γ . Each cusp ζ of Γ may be given by $\zeta = A^{-1}\infty$ where $A = \begin{pmatrix} * & * \\ a_1 & a_2 \end{pmatrix}$ is real, $|A| = 1$, $A = \text{unit}$

matrix I if $\zeta = \infty$. The cyclical group of the transformations of $\overline{\Gamma}$ with the fixed point ζ is generated by the transformation with the "principal matrix" of ζ :

$$(2.1) \quad P = A^{-1} U^N A \quad (U^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ for real } \lambda, U^1 = U)$$

where $N^* = N_A^*$ denotes a certain positive number; we shall write N for N_I^* .

Next, if $2 \leq r \leq r_0$, $r_0 - 2$ being sufficiently small, we determine arbitrarily a system of multipliers v_r on Γ of modulus 1, belonging to the dimension $-r$, which depends continuously on r and satisfies the conditions $[A, B, C]$ of [1]; for $r = 2$, v_r coincides with the preassigned v_2 . We denote by $K_r = \{\Gamma, -r, v_r\}$ the "class" of automorphic forms corresponding to $\overline{\Gamma}, -r, v_r$, by K_r^+ the linear manifold of the integral forms in K_r vanishing in all cusps of $\overline{\Gamma}$. For each cusp ζ of $\overline{\Gamma}$ we put

$$(2.2) \quad v_r(P) = \exp 2\pi i k^*(r) \text{ where } 0 \leq k^*(r) < 1 \text{ (if } 2 < r \leq r_0)$$

and $k^*(r) = k_A^*(r)$ is a continuous function of r in $2 \leq r \leq r_0$; we shall write $k(r)$ for $k_I^*(r)$.

In order to deal with Poincaré series we may fix the notation in such a manner that each cusp ζ of Γ is represented by

only one A and that either any two such cusps, represented by two of the matrices I, A, B , are incongruent to each other mod $\bar{\Gamma}$ or the corresponding two matrices coincide. For the matrices

$M = AL \in A\Gamma$ we define $v_r(M) = v_r(A)\sigma^{(r)}(A, L)v_r(L)$ where $v_r(A)$ denotes, if $A \neq I$, an arbitrary continuous function of r in

$2 \leq r \leq r_0$, throughout of modulus 1. Now, ν being an integer, we introduce the Poincaré series associated to the cusp ζ of $\bar{\Gamma}$ by

$$(2.3) \quad G(\tau, A, K_r, \nu + k^*(r)) = \sum_{M \in Z(A, \Gamma)} v_r^{-1}(M) (m_1 \tau + m_2)^{-r} \\ \cdot \exp 2\pi i \frac{\nu + k^*(r)}{N^*} M\tau$$

where $r > 2$ and $M = \begin{pmatrix} * & * \\ m_1 & m_2 \end{pmatrix}$ runs over a complete system

$Z(A, \Gamma)$ of matrices of $A\Gamma$ with different second rows (m_1, m_2) .

The powers used here are principal values $(-\pi < \arg < +\pi)$. Every G represents an automorphic form of K_r , analytic in $y > 0$. For $\nu + k^*(r) > 0$, G belongs to K_r^+ . Evidently, every G is a continuous function both of τ and r for $y > 0$, $2 < r \leq r_0$. In the following we consider only the case

$$(2.4) \quad \nu + k^*(2) < 0, \text{ i. e. } \nu \leq -1 \text{ if } 0 \leq k^*(2) < 1; \nu \leq -2 \\ \text{if } k^*(2) = 1.$$

Applying the results of [3] and assuming $2 < r \leq r_0$ as well as (2.4), we obtain

THEOREM 1. (a) $\mathfrak{G}(\tau, A, K_r, \nu+k^*(r))$ vanishes in all cusps of $\overline{\Gamma}$ which are not congruent to $\xi = A^{-1} \infty \pmod{\overline{\Gamma}}$. (b) In $y > 0$ an expansion

$$(2.5) \quad \nu_r(A)(a_1\tau+a_2)^r \mathfrak{G}(\tau, A, k_r, \nu+k^*(r)) \\ = 2t_A^{\nu+k^*(r)} + \sum_{n+k^*(r)=0} \beta_n t_A^{n+k^*(r)}$$

holds where $t_A^\lambda = \exp 2\pi i \frac{\lambda}{N} A\tau$ when λ is real

($t_A = t_A^1$ being the locally uniformizing variable to ξ)

and the β_n denote constants. (c) In the sense of the

metrization of the automorphic forms of K_r ,

$\mathfrak{G}(\tau, A, K_r, \nu+k^*(r))$ is orthogonal to all forms of K_r^+ .

(d) As an automorphic form of K_r , $\mathfrak{G}(\tau, A, K_r, \nu+k^*(r))$

is uniquely determined by (a), (b), (c).

The result of the summation method, when applied to the series (2.3), consists in

THEOREM 2. On every compact domain in $y > 0$,

$G(\tau, A, K_r, \nu+k^*(r))$ tends, as $r \rightarrow 2+0$, uniformly to an automorphic form $G(\tau, A, K_2, \nu+k^*(2))$ of K_2 which is analytic in $y > 0$ and satisfies the conditions (a) (b) (c) of theorem 1 in the case $r = 2$. By these conditions, $G(\tau, A, K_2, \nu+k^*(2))$, as an automorphic form of K_2 , is uniquely determined.

3. Next, we have to use a reduction theorem. We fix on any canonical fundamental domain K_Γ of $\overline{\Gamma}$ and denote the cusps of K_Γ by ζ_h ($1 \leq h \leq \sigma_0$). Corresponding to (2.1, 2), we put, whenever $2 \leq r \leq r_0$:

$$\zeta_h = A_h^{-1} \infty \text{ with a real matrix } A_h = \begin{pmatrix} * & * \\ a_{h1} & a_{h2} \end{pmatrix}, |A_h| = 1; \quad (3.1)$$

$$k_h(r) = k_{A_h}^*(r) \quad (0 \leq k_h(r) < 1 \text{ if } 2 < r \leq r_0).$$

For brevity we write

$$(3.2) \quad G_{-r}(\tau, A, \nu) = G(\tau, A, K_r, \nu+k^*(r)) \quad (2 \leq r \leq r_0).$$

Then the reduction theorem to be applied here says

THEOREM 3. Let $F(\tau)$ be a non-integral automorphic form of K_2 , analytic in $y > 0$. There is one, and only one, decomposition $F(\tau) = \Lambda(\tau) + \phi(\tau)$ where $\phi(\tau)$ denotes an integral form of K_2 , $\Lambda(\tau)$ a linear combination of a finite number from among the $\mathfrak{G}_{-2}(\tau, A_h, \nu)$ ($1 \leq h \leq \sigma_0$, ν an integer, $\nu + k_h(2) < 0$) with constant coefficients. These coefficients are uniquely determined by the principal parts of $F(\tau)$ in the ζ_h .

In order to calculate the coefficients, we write in the general case (see (2.1, 2))

$$(3.3) \quad v_2(A)(a_1\tau + a_2)^2 F(\tau) = \sum_{n \geq -n_A} b_n(A, F) t_A^{n+k^*(2)}.$$

Then by theorems 1 and 2 we obtain in the notation of (3.1, 2, 3):

$$(3.4) \quad \Lambda(\tau) = \frac{1}{2} \sum_{h=1}^{\sigma_0} \sum_{\substack{\nu + k_h(2) < 0 \\ \nu \geq -n_h}} b_{\nu}(A_h, F) \mathfrak{G}_{-2}(\tau, A_h, \nu),$$

n_A and n_h , as later on n_0 , denoting suitable integers. On the other hand, we introduce the Fourier coefficients of an arbitrary

automorphic form $f(\tau)$ of K_2 . Using $N = N_I^*$, $k(r) = k_I^*(r)$, we write

$$(3.5) \quad f(\tau) = \sum_{n=-n_0}^{\infty} b_n(f) t_I^{n+k(2)} \quad (t_I = \exp 2\pi i \frac{\tau}{N}).$$

The main theorem to be proved here is essentially based on the explicit form of the Fourier coefficients of the series (2.3).

Putting

$$(3.6) \quad G_{-r}(\tau, A, \nu) = 2 \delta_{A, I} t^{\nu+k(r)} + \sum_{n+k(r)>0} c_{-r}(n, A, \nu) t^{n+k(r)} \quad (r > 2)$$

and, for brevity, $k = k(r)$, $k^* = k^*(r)$, we find, when $r > 2$,

$n + k > 0$:

$$(3.7) \quad c_{-r}(n, A, \nu) = i^{-r} \frac{4\pi}{N} \left(\frac{n+k}{N}\right)^{\frac{1}{2}(r-1)} \left(\frac{-\nu-k^*}{N^*}\right)^{-\frac{1}{2}(r-1)} \\ \times \sum_{m_1 \in T^+(A, \Gamma)} m_1^{-1} W_{m_1}^{(r)}(n, A, \nu) I_{r-1} \left(\frac{4\pi}{m_1} \sqrt{\frac{(n+k)(-\nu-k^*)}{NN^*}} \right)$$

Here, and with Bessel functions $I_{r-1}(z)$, all powers with non-integral exponents are to be taken as principal values ($|\arg| < \pi$). $T^+(A, \Gamma)$

denotes the set of the numbers $m_1 > 0$ appearing in the second rows of the matrices $M = \begin{pmatrix} * & * \\ m_1 & m_2 \end{pmatrix}$ of AI . Moreover

$$W_{m_1}^{(r)}(n, A, \nu) = \sum_Q \nu_r^{-1}(Q) \exp 2\pi i \left\{ \frac{(n+k)j}{m_1 N} + \frac{(\nu+k^*)j}{m_1 N^*} \right\}$$

where $Q = \begin{pmatrix} j' & * \\ m_1 & j \end{pmatrix}$, when $m_1 \in T^+$ is fixed, runs over a full system of matrices of $Z(A, I)$ whose elements j are incongruent to each other (mod $m_1 N$). The expansions (3.3, 5, 6) are valid in the total domain $y > 0$.

From theorem 2, it follows that

$$\lim_{r \rightarrow 2+0} c_{-r}(n, A, \nu) = c_{-2}(n, A, \nu)$$

exists whenever $n + k(r) > 0$ for $2 < r \leq r_0$; besides,

$$c_{-2}(0, A, \nu) = 0 \text{ if } k(2) = 0.$$

Theorem 3 and (3.4) yield the important formula

$$(3.8) \quad b_n(F) = \frac{1}{2} \sum_{h=1}^{\sigma_0} \sum_{\substack{\nu+k_h(2)<0 \\ \nu \geq -n_h}} b_\nu(A_h, F) c_{-2}(n, A_h, \nu) + b_n(\phi).$$

4. In order to prove an asymptotic formula for the $b_n(F)$ ($n + k(2) > 0$), we shall try to decompose the coefficients c_{-2} of (3.8) into two terms, a principal term and a remainder. The principal term is obtained from (3.7) by putting $r = 2$ and then omitting the terms with $m_1 > \text{const} \sqrt{n}$ from the \sum_{m_1} . The main difficulty consists of course in finding an estimation of the remainder.

We next introduce auxiliary functions by means of the following definition: Let λ be a real number, $S = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ a real matrix of determinant 1, ρ a real number or $\rho = \infty$. Then we define

$$\exp 2\pi i \lambda S \rho = \exp 2\pi i \lambda S(\rho) = 0 \text{ if } S\rho = \infty \text{ (i. e. } \rho = -\frac{d}{c}\text{)}$$

and, whenever $r > 2$:

$$(4.1) \quad E(\tau, \rho; A, K_r, \nu+k^*) = \sum_{M \in Z(A, \Gamma)} v_r^{-1}(M) (m_1 \tau + m_2)^{-r} \exp 2\pi i \frac{\nu+k^*}{N^*} M \rho.$$

This series represents a function analytic in $y > 0$ and satisfying the following functional equations: If $L = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma$ then

$$(4.2) \quad E(L\tau, L\rho; A, K_r, \nu+k^*) = v_r(L) (\gamma\tau + \delta)^r E(\tau, \rho; A, K_r, \nu+k^*);$$

if S is taken as above, then

$$(4.3) \quad E(S\tau, S\rho; A, K_r, \nu+k^*) = V_r(A, S)(c\tau+d)^r E(\tau, \rho; AS, K'_{r,S}, \nu+k^*)$$

where $V_r(A, S)$ depends only on r, A, S and is a continuous function of r of modulus 1 in $2 \leq r \leq r_0$. $K'_{r,S}$ denotes the class

$$K'_{r,S} = \{l'_S, -r, \nu'_{r,S}\}, \text{ where } \Gamma'_S = S^{-1}\Gamma S$$

and $\nu'_{r,S}$ is a system of multipliers, obtained from ν_r by a certain process of transformation.

The series (4.1) provides a certain approximation to the corresponding G_{-r} . We consider the difference

$$(4.4) \quad D_{-r}(\tau, A, \nu) = G_{-r}(\tau, A, \nu) - E(\tau, \infty; A, K_r, \nu+k^*(r)) \quad (r > 2)$$

on the vertical half-strip V , defined by $|x| \leq C, y \geq \alpha$ ($C > 1, \alpha > 0$). Subtracting term by term, we see that $D_{-r}(\tau, A, \nu)$ has the majorant

$$(4.5) \quad \sum_{\substack{M \in Z(A, \Gamma) \\ m_1 \neq 0}} C_1 |m_1|^{-1} |m_1 \tau + m_2|^{-r-1}$$

on V where C_1 (as in the following C_2, C_3, \dots) is a constant.

The series (4.5) is uniformly convergent on V for $r > 0$.

Moreover, applying (4.2) in the case $\rho = \infty, L = U^N$ and

using the properties of the general series (4.1) in any vertical half-strip similar to V , we obtain a Fourier expansion of the form

$$E(\tau, \infty; A, K_r, \nu + k^*(r)) = \sum_{n+k(r)>0} e_{-r}(n, A, \nu) t^{n+k(r)} \quad (r > 2)$$

valid for $y > 0$ where

$$(4.6) \quad e_{-r}(n, A, \nu) = \frac{2(-2\pi i)^r}{N^r \Gamma(r)} (n+k)^{r-1} \sum_{m_1 \in T^+(A, \Gamma)} m_1^{-r} W_{m_1}^{(r)}(n, A, \nu).$$

This expression may be derived from (3.7) by replacing each $I_{r-1}(z)$ by the first term of the power series of that function, i. e. by $\Gamma^{-1}(r) \left(\frac{1}{2}z\right)^{r-1}$. Corresponding to (4.4), we write when $r > 2$, $n + k(r) > 0$:

$$(4.7) \quad c_{-r}(n, A, \nu) = e_{-r}(n, A, \nu) + d_{-r}(n, A, \nu).$$

Now we combine theorem 2 with the results on the series

D_{-r} and we find

THEOREM 4. On every compact domain in $y > 0$, $E(\tau, \infty; A, K_r, \nu + k^*(r))$ tends, as $r \rightarrow 2+0$, uniformly to a limiting function $E_{-2}(\tau, A, \nu)$ which is analytic in $y > 0$. This function admits a Fourier expansion of the form (3.5) with the coefficients

$$b_n(E_{-2}) = \lim_{r \rightarrow 2+0} e_{-r}(n, A, \nu) = e_{-2}(n, A, \nu) \\ (n + k(2) \geq 0).$$

We have

$$c_{-2}(n, A, \nu) = e_{-2}(n, A, \nu) + d_{-2}(n, A, \nu) \\ (n + k(2) \geq 0)$$

where $d_{-2}(n, A, \nu)$ can be represented by an absolutely convergent series.

The series representing $d_{-r}(n, A, \nu)$ is obtained by replacing $I_{r-1}(z)$, whenever it occurs in (3.7), by $I_{r-1}(z) - \Gamma^{-1}(r)(\frac{1}{2}z)^{r-1}$.

5. We are now in a position to describe the procedure which leads to asymptotic formulae for the $b_n(F)$ ($n \rightarrow \infty$). We put

$$\mu_{n,\nu} = \mu_{n,\nu}(A, \Gamma) = 4\pi a_\nu(A, \Gamma) \sqrt{\frac{(n+k(2))(-\nu-k^*(2))}{NN^*}} > 0$$

where $a_\nu(A, \Gamma)$ denotes an arbitrary positive number depending only on ν, A, Γ . With respect to (3.8), the three parameters ν, A, Γ may be regarded as fixed. By

$$c_{-r}^{(1)}, e_{-r}^{(1)}, d_{-r}^{(1)} \text{ resp. } c_{-r}^{(2)}, e_{-r}^{(2)}, d_{-r}^{(2)} \quad < r \leq r_0,$$

we denote the expressions arising from (3.7), (4.6) and the difference of these series by the additional conditions $m_1 \leq \mu_{n, \nu}$ or $m_1 > \mu_{n, \nu}$ resp. in Σ_{m_1} . Plainly, the "principal terms" $c_{-r}^{(1)}$, $e_{-r}^{(1)}$, $d_{-r}^{(1)}$, involving only finite sums, tend, as $r \rightarrow 2 + 0$, to limits which may easily be written down explicitly. Using theorems 3 and 4, we see that the "remainders" $c_{-r}^{(2)}$, $e_{-r}^{(2)}$, $d_{-r}^{(2)}$ also tend to certain limits if $r \rightarrow 2 + 0$. We indicate the six limits by changing the suffix $-r$ into -2 .

Now, inserting $c_{-2} = c_{-2}^{(1)} + c_{-2}^{(2)}$ in (3.8), we obtain a decomposition of $b_n(F)$ in the form $b_n(F) = H_n(F) + R_n(F)$ where $H_n(F)$ arises from the double sum in (3.8) by the substitution $c_{-2} \rightarrow c_{-2}^{(1)}$ and $R_n(F)$ is given by

$$(5.1) \quad R_n(F) = \frac{1}{2} \sum_{h=1}^{\sigma_0} \sum_{\substack{\nu + k_h(2) < 0 \\ \nu \geq n_h}} b_\nu(A_h, F) c_{-2}^{(2)}(n, A_h, \nu) + b_n(\phi).$$

The estimation of $R_n(F)$ to be proved later on rests upon the decomposition formulae

$$(5.2) \quad c_{-2}^{(2)} = e_{-2}^{(2)} + d_{-2}^{(2)} = e_{-2} - e_{-2}^{(1)} + d_{-2}^{(2)}$$

and upon an estimation of $b_n(\phi)$. The result will be

$$(5.3) \quad R_n(F) = O(n^{\frac{3}{2}}) \quad (n \rightarrow \infty).$$

Considering (5.3), we add some remarks. If Γ is a congruence subgroup of the modular group and χ_2 a congruence character, then, by aid of A. Weil's estimation of certain exponential sums, even $R_n(F) = O(n^{\frac{3}{4} + \varepsilon})$ can be proved for every $\varepsilon > 0$. The relative exactness of both estimations is very high, as the single terms in the numbers $c_{-2}^{(1)}$, at least in the most important cases, tend to infinity like $e^{\gamma_n \sqrt{n}}$ where γ_n has a positive lower bound. The principal term $H_n(F)$ corresponds to that part of Rademacher's partition series which contains, according to the customary interpretation, the asymptotic expansion of the partition function.

6. For $m_1 \in T^+(A, \Gamma)$ we denote by $\lambda_{m_1}(A, \Gamma)$ the (finite) number of the real numbers m_2 in $M = \begin{pmatrix} * & * \\ m_1 & m_2 \end{pmatrix} \in A\Gamma$ which are incongruent to each other (mod $m_1 N$). Then we have

$$(6.1) \quad |e_{-2}^{(1)}(n, A, \nu)| \leq C_2(n + k(2)) \sum_{\substack{m_1 \in T^+(A, \Gamma) \\ m_1 \leq n, \nu}} m_1^{-2} \lambda_{m_1}(A, \Gamma),$$

$$|d_{-2}^{(2)}(n, A, \nu)| \leq C_3(n + k(2))^2 \sum_{\substack{m_1 \in T^+(A, \Gamma) \\ m_1 > \mu, \nu}} m_1^{-4} \lambda_{m_1}(A, \Gamma).$$

The method used in [4] in order to discuss the convergence of Poincaré's series may also be applied to estimate the sums in (6.1). Thus, if A, ν are fixed, we obtain

$$(6.2) \quad e_{-2}^{(1)}(n, A, \nu) = O(n \log n), \quad d_{-2}^{(2)}(n, A, \nu) = O(n) \quad (n \rightarrow \infty).$$

The estimation of e_{-2} is more difficult. Putting $Q = A_h L$
 $= \begin{pmatrix} * & * \\ q_1 & q_2 \end{pmatrix} (1 \leq h \leq \sigma_0, L \in \Gamma)$ we find, using (4.2, 3), when $r > 2$:

$$E(\tau, \infty; A, K_r, \nu + k^*(r)) = V_r'(A, A_h) V_r^{-1}(Q) (q_1 \tau + q_2)^{-r} \times \\ \times \{E(Q\tau, \infty; A A_h^{-1}, K_r', A_h^{-1}, \nu + k^*(r)) + \Delta_Q^{(r)}(Q\tau)\}$$

where $V_r'(A, A_h)$ is of modulus 1 and $\Delta_Q^{(r)}(\tau)$ the difference of two series of the type (4.1). Subtracting here again term by term, it is seen that $\Delta_Q^{(r)}(\tau)$ tends to a limit $\Delta_Q^{(2)}$ as $r \rightarrow 2 + 0$ and that $|\Delta_Q^{(r)}(Q\tau)| \leq C_4 q_1$ for $2 \leq r \leq r_0$. From this inequality follows by (6.3)

$$(6.4) \quad |E(\tau, \infty; A, K_r, \nu + k^*(r))| \leq C_5 |q_1| |q_1 \tau + q_2|^{-r} \quad (2 \leq r \leq r_0).$$

Next, in order to estimate $e_{-2}(n, A, \nu)$, one has to consider the integral representation of e_{-2} . We divide the straight segment of integration into parts, each contained in one cusp sector of the network of the fundamental domains LK_Γ ($L \in \Gamma$), and apply (6.4) to the integrand on that part. Then, finally, we obtain (A, ν being fixed)

$$(6.5) \quad e_{-2}(n, A, \nu) = O(n^{\frac{3}{2}}) \quad \text{when } n \rightarrow \infty.$$

By a similar argument it follows that

$$(6.6) \quad b_n(\phi) = O(n \log n) \quad \text{when } n \rightarrow \infty.$$

Now, (5.3) follows from (6.2, 5, 6). The final result is

THEOREM 5. (For abbreviations see (3.1, 3, 5, 7), theorem 3 and §5.)

$$b_n(F) = \frac{1}{2} \sum_{h=1}^{\sigma_0} \sum_{\substack{\nu+k_h(2)<0 \\ \nu \geq -n_h}} b_{\nu}(A_h, F) c_{-2}^{(1)}(n, A_h, \nu) + O(n^{\frac{3}{2}}) \quad (n \rightarrow \infty).$$

The constants $a_\nu(A_h, \Gamma)$ may be fixed in such a manner

that the arguments of Bessel's functions I_1 , occurring in the $c_{-2}^{(1)}(n, A_h, \nu)$, have the same lower bound.

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I shall in the following give a brief sketch of an independent approach to certain methods and results that I have earlier [1] indicated as part of a more general theory.

1. Let us consider a bounded domain B in the space of n complex variables z_1, \dots, z_n , where in the following, for brevity, we will write z for the n -tuple z_1, \dots, z_n . We assume that we have a group G of regular analytic mappings $z \rightarrow gz$ of B onto itself, which acts transitively on B . Let $j_g(z)$ denote the Jacobian of gz with respect to z , and let $k(z, \zeta)$ be the Bergman kernel-function of the domain B . We have then

$$(1.1) \quad j_g(z) \overline{j_g(\zeta)} k(gz, g\zeta) = k(z, \zeta),$$

further that the volume element $d\omega_z = k(z, z)dz$, (where by dz we denote the euclidean volume element), is invariant under G .

Consider the Hilbert space of analytic functions $f(z)$ in B for which

$$(1.2) \quad \int_B \frac{|f(z)|^2}{(k(z, z))^r} d\omega_z < \infty,$$

where r is real and large enough that our Hilbert space is not empty; this is, for instance, the case if $r \geq 1$. Constructing a complete set of orthonormal functions $\phi_i^{(r)}(z)$, $i = 1, 2, \dots$, and forming the generalized kernel function

$$(1.3) \quad k_r(z, \zeta) = \sum_i \phi_i^{(r)}(z) \overline{\phi_i^{(r)}(\zeta)},$$

one finds that

$$(1.4) \quad (j_g(z))^r \overline{(j_g(\zeta))^r} k_r(gz, g\zeta) = k_r(z, \zeta).$$

From (1.1) and (1.4) and the fact that G acts transitively on B , one can show that

$$(1.5) \quad k_r(z, \zeta) = c(r) (k(z, \zeta))^r,$$

where the value of $c(r)$ is given by the integral

$$(1.6) \quad \frac{1}{c(r)} = \int_B \frac{|k(z, \zeta)|^{2r}}{(k(z, z)k(\zeta, \zeta))^r} d\omega_\zeta.$$

For functions in our Hilbert space (1.2), we then have

$$(1.7) \quad f(z) = c(r) \int_B \left(\frac{k(z, \zeta)}{k(\zeta, \zeta)} \right)^r f(\zeta) d\omega_\zeta.$$

We can show that (1.7) also holds for the class of functions $f(z)$

for which $(k(z, z))^{-\frac{r}{2}} f(z)$ is bounded in B , provided

$$(1.8) \quad \int_B \frac{1}{(k(\zeta, \zeta))^{r/2}} d\omega_{\zeta} < \infty,$$

therefore certainly for $r \geq 2$.

2. Let now Γ be a discrete subgroup of G and let us first assume that the fundamental domain D of Γ in B is compact. Further, denoting the elements of Γ by γ , let us assume that we have a representation of Γ by unitary $\nu \times \nu$ matrices $\chi(\gamma)$, and let $F(z)$ be a column-vector whose ν components are analytic functions regular in the interior of B , and satisfying the relation

$$(2.1) \quad F(\gamma z) = \chi(\gamma)(j_{\gamma}(z))^{-r} F(z),$$

where r is an integer ≥ 2 , or more generally a number for which (1.8) is satisfied, and for which $(j_g(z))^r$ is single valued on the group G .

We have then

$$F(z) = c(r) \int_B \left(\frac{k(z, \zeta)}{k(\zeta, \zeta)} \right)^r F(\zeta) d\omega_{\zeta},$$

from which we get

$$(2.2) \quad F(z) = c(r) \int_D \frac{K_r(z, \zeta; \chi)}{(k(\zeta, \zeta))^r} F(\zeta) d\omega_\zeta,$$

where

$$(2.3) \quad K_r(z, \zeta; \chi) = \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)} (j_\gamma(z))^r (k(\gamma z, \zeta))^r.$$

The operator occurring on the right-hand side in (2.2) is seen to be hermitian; it has only the functions $F(z)$ as eigenfunctions and each of these with eigenvalue 1. Denoting the number of linearly independent such functions by N_r , we have, therefore,

$$N_r = c(r) \int_D \frac{\sigma(K_r(z, z; \chi))}{(k(z, z))^r} d\omega_z,$$

where σ denotes the trace of the matrix K_r . Introducing here the series (2.3) and combining terms where the γ 's are conjugate to each other with respect to Γ , we get

$$(2.4) \quad N_r = c(r) \sum_{\{\gamma\}_\Gamma} \sigma(\overline{\chi(\gamma)}) \int_{D_\gamma} \left(\frac{k(\gamma z, z)}{k(z, z)} \right)^r (j_\gamma(z))^r d\omega_z$$

where $\{\gamma\}_\Gamma$ indicates that we sum over one representative of each conjugate class, and D_γ denotes the fundamental domain in B of Γ_γ , the subgroup of elements of Γ that commute with γ . A somewhat more extensive transformation of the terms on the right-hand side of (2.4) is indicated in [1].

Let us first see what we can conclude about N_r without making further assumptions about B or G . Writing

$$\frac{k(\gamma z, z)}{k(z, z)} j_\gamma(z) = \frac{k(\gamma z, z)}{(k(\gamma z, \gamma z)k(z, z))^{1/2}} \frac{j_\gamma(z)}{|j_\gamma(z)|},$$

we see that for large r the dominant contribution to the right-hand side of (2.4) will come from the γ 's that have fixed points.

Denoting by ρ_i a set of representatives (it will be finite) of the conjugate classes with fixed points, that are different from the identity, we can obtain an asymptotic expression for N_r :

$$(2.5) \quad N_r = \nu V(D) P_0(r) + \sum_i \sigma(\chi(\rho_i))' V(G_{\rho_i} / \Gamma_{\rho_i}) P_{\rho_i}(r) \cdot \epsilon_{\rho_i}^r + O\left(\frac{1}{r}\right),$$

where $V(D)$ denotes the volume of D , $P_0(r)$ is a polynomial of degree n in r , $P_{\rho_i}(r)$ is a polynomial depending on ρ_i whose degree m is equal to the complex dimension of the set of fixed points of ρ_i , ϵ_{ρ_i} is the value of $j_{\rho_i}(z_{\rho_i})$ at a fixed point z_{ρ_i} of ρ_i , $V(G_{\rho_i} / \Gamma_{\rho_i})$ measures the volume of the fundamental domain of Γ_{ρ_i} in G_{ρ_i} (where G_{ρ_i} denotes the subgroup of elements of G that commute with ρ_i). In particular if ρ_i has an isolated fixed point, we have

$$V(G_i / \Gamma_i) P_i(r) = \frac{1}{m_i |E_n - J_{P_i}(z_{P_i})|},$$

where m_i is the order of the finite group Γ_i , E_n is the $n \times n$ identity matrix, and $J_{P_i}(z_{P_i})$ is the Jacobi matrix at a fixed point, and $| \quad |$ denotes the determinant.

Since N_r is necessarily an integer it is easy to conclude that from a certain r on, the formula (2.5) will be exact without the remainder term $O(\frac{1}{r})$. It seems difficult, however, to prove that it is exact without the O -term for all $r \geq 2$, without some additional assumption.

If, however, we restrict ourselves to the symmetric domains B , we can verify directly that $c(r)$ is a polynomial $P_0(r)$, that the γ 's without fixed points give no contribution to the right-hand side of (2.4), and that the contribution of the γ 's with fixed points is such that (2.5) is exact for $r \geq 2$, without the O -term. For reasons apparent from [1], one need only consider the irreducible symmetric spaces, and I have verified it for three of the four main types of these (the types called I, II, III in [2], §48).

If, restricting ourselves to the case that B is symmetric, we apply the same method for the case that the fundamental domain D is not compact, but still has finite volume, we meet great

technical difficulties. These however seem surmountable so long as one supposes that the "non-compact parts" of D are of the types that occur for the groups that come from the arithmetic of quadratic forms. In the simple case that B is a product of n unit circles, and the fundamental domain D is only allowed non-compact parts like those that occur for the so-called Hilbert modular groups, the formula (2.5) has been shown to be valid if we add some terms coming from aggregates of elements γ without fixed points in B , but which leave fixed some part of the "non-compact boundary" of D .

As I have indicated in [1], the method may be extended also to the determination of the traces of certain types of operators, that carry the space of forms $F(z)$ satisfying (2.1) into itself, as typified by the so-called Hecke operators.

3. We may generalize somewhat the approach we have outlined so that it covers the case that B is the full complex space and Γ a group of motions with compact fundamental domain. The main interest in this lies in the unification, but some of the resulting formulas are new even in this case.

Let us drop the requirement that B be bounded, and suppose that we have a function $p_g(z)$ on G and B , which is analytic in z throughout B ; for fixed z in B it should be bounded, and finally satisfy the relation

$$(3.1) \quad |p_{g'}(gz)p_g(z)| = |p_{g'g}(z)|,$$

for all g' and g in G and z in B . Further assume that the Hilbert space

$$(3.2) \quad \int_G |p_g(z)|^2 |f(gz)|^2 dg < \infty,$$

where dg is the left-invariant Haar measure on G , is non-empty.

By taking a complete set of orthonormal functions, we define a function $k^{(p)}$ with the property

$$p_g(z) \overline{p_g(\zeta)} k^{(p)}(gz, g\zeta) = k^{(p)}(z, \zeta).$$

Denoting by $d\omega_z$ an invariant element of volume, we consider the general Hilbert space for $r \geq 1$

$$(3.3) \quad \int_B \frac{|f(z)|^2}{|k^{(p)}(z, z)|^r} d\omega_z < \infty,$$

and construct a complete set of orthonormal functions, and then

define $k_r^{(p)}(z, \zeta)$ in a way corresponding to (1.3). We obtain again

$$k_r^{(p)}(z, \zeta) = c(r)(k^{(p)}(z, \zeta))^r.$$

We can now repeat our arguments of paragraph 2, with $p_\gamma(z)$ taking the place of $j_\gamma(z)$, provided it is possible (by eventually multiplying the $p_\gamma(z)$ with suitable factors of absolute value one, depending on γ only) to obtain

$$(3.4) \quad p_{\gamma'}(\gamma z) p_\gamma(z) = p_{\gamma' \gamma}(z),$$

for all γ' and γ in Γ and z in B .

If B is complex n -space and \mathfrak{G} the group of translations $z \rightarrow z+g$, where z and g are column vectors with n complex components, we put

$$(3.5) \quad p_g(z) = e^{-\bar{g}' Q z - \frac{1}{2} \bar{g}' Q g},$$

where Q is a Hermitian positive definite $n \times n$ matrix. Then (3.1) holds and the Hilbert space defined by (3.2) is non-empty; also $p_g(z)$ is bounded for fixed z . The problem of satisfying (3.4) for our discrete group Γ with some choice of Q , leads to the usual

arithmetical condition that the period matrix has to satisfy in order that the field of abelian functions should be non-degenerate. This approach also works if there are motions other than translations in Γ as long as the fundamental domain is compact.

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Seminar V. ANALYTIC FUNCTIONS
AS RELATED TO BANACH ALGEBRAS

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1. σ -BANACH ALGEBRAS

Let A be a linear algebra over the complex numbers, and let there be a family of semi-norms

$$||a||_1 \leq ||a||_2 \leq \dots$$

each satisfying also $||ab|| \leq ||a|| ||b||$. Then A is a σ -normed algebra. If it is complete, it is a σ -Banach algebra. Henceforth we deal only with commutative σ -Banach algebras with unit.

Let Δ_r be the class of homomorphisms ζ of A onto the complex numbers C satisfying

$$1.1 \quad |\zeta(a)| \leq ||a||_r.$$

Then $\Delta = \Delta_1 \cup \Delta_2 \cup \dots$ is the class of all continuous homomorphisms of A on C . A fact (presented elsewhere) is that

1.11 if a_1, \dots, a_n have no common 0 on Δ , then

$$a_1 b_1 + \dots + a_n b_n = 1 \text{ for some } b_1, \dots, b_n \text{ in } A.$$

For a_1, \dots, a_n in A , let $\sigma(a_1, \dots, a_n)$ and $\sigma_r(a_1, \dots, a_n)$ be the class of all $\lambda_1, \dots, \lambda_n$ such that $a_1 - \lambda_1, \dots, a_n - \lambda_n$ have a

common zero on Δ (on Δ_r , resp.). From [1] (see the references) it can be deduced that, for A semi-simple, if ϕ is holomorphic on a neighborhood of Δ , then ϕ belongs to A , in the sense that

$$\zeta(a) = \phi(\zeta(a_1), \dots, \zeta(a_n))$$

for some a in A . When $n = 1$, the semi-simplicity hypothesis is dispensable.

Of course, σ may be open. In fact, if it is not, and $n = 1$, there are topological zero divisors ([4], §11.8). In fact, more is true (when $n = 1$).

1.2 If σ_n is not contained in the interior of some σ_p ($p > n$), then $a_1 - \lambda_1$ is a topological zero divisor for some λ_1 .

When $n > 1$ a similar result holds if the Shilov boundaries of $\sigma_1, \sigma_2, \dots$ have a common point (see 3.6 below).

1.3 THEOREM: Let A be a σ -Banach algebra with one rational generator ¹⁾ z such that $z - \lambda$ is never a

1) z_1, \dots, z_n are rational generators if the quotients pq^{-1} , p, q polynomials in z_1, \dots, z_n , q non-singular, are dense in A .

topological zero divisor. Then A is the vector space direct sum of two closed subalgebras H and J , where H is the algebra of functions holomorphic on an open set and J is the radical.

$\sigma(z)$ is the open set in this case. The converse holds, of course:

1.4 The σ -Banach algebra $\text{Hol}(\Omega)$ of functions holomorphic on a separable (connected) analytic manifold Ω has no topological zero divisors, except 0.

To prove this one chooses K_1, K_2, \dots compact in Ω , such that their interiors cover Ω , and defines $\|f\|_n = \max|f(K_n)|$. Now let $f \neq 0$. Using a weak form of Weierstrass' preparation theorem, one can enlarge each K_n slightly so that f has no zeros on its Shilov boundary. This gives an equivalent system of semi-norms and $\|fg\|_n \geq k_n \|g\|_n$ for every g in $\text{Hol}(\Omega)$.

Every element in the radical of any σ -Banach algebra is a topological zero divisor.

When z_1, \dots, z_n are rational generators then $\sigma(z_1, \dots, z_n) = \Delta$, essentially. Even for $n = 1$ we can have $\sigma(z)$ open, but the

representing functions need not all be holomorphic on $\sigma(z)$.

Whether or not the algebra has such a representation as in

1.3, some classical results can be generalized:

1.5 THEOREM: Let A_1, A_2, \dots be ideals in A such that the hull of A_n does not meet ²⁾ Δ_n . Then there is a non-zero element common to the ideals $A_1, A_1 A_2, A_1 A_2 A_3, \dots$.

Here is a special case:

1.51 THEOREM: Let $\zeta_1, \zeta_2, \dots \in \Delta$ with only finitely many in each Δ_n , and none in some Δ_m . Let μ_1, μ_2, \dots be positive integers, and $\varepsilon > 0$. Then there is an $f \in A$ such that $\|f-1\|_m < \varepsilon$ and f vanishes to the μ_i -th order at ζ_i ($i = 1, 2, \dots$).

(This means $f \in M^{\mu_i}$ where M is the kernel of ζ_i .)

A Mittag-Leffler form is also possible.

1.52 THEOREM: Consider the hypotheses of 1.51.

2) This means for each $\zeta \in \Delta_n$ there is an $a \in A_n$ such that $\zeta(a) \neq 0$.

Let there be given also $g_1, g_2, \dots \in A$. Then we can find $g \in A$ such that $g - g_i$ vanishes to the μ_i -th order at ζ_i ($i = 1, 2, \dots$).

2. REPRESENTING MAXIMAL IDEALS BY POINTS

A Bereich Ω is an n -dimensional analytic manifold with n functions z_1, \dots, z_n such that the mapping $z : \Omega$ into C^n , where ω goes into $z_1(\omega), \dots, z_n(\omega)$, is a local analytic isomorphism.

Ω is schlicht if z is 1-1; one can then say Ω is contained in C^n . As already indicated, $\text{Hol}(\Omega)$ is a σ -Banach algebra. The following theorem is based almost wholly on the ideas of H. Cartan.

2.1 THEOREM: Let Ω be schlicht and maximal, in the sense that if $\Omega_1 \supset \Omega$ then some f in $\text{Hol}(\Omega)$ cannot be extended to Ω_1 . Then $\Delta = \Omega$.

Proof: As a metric in Ω use $\max_{1 \leq i \leq n} |\lambda_i - \mu_i|$ as the distance from $\lambda_1, \dots, \lambda_n$ to μ_1, \dots, μ_n . Let K be compact, and let Δ_K be the $\zeta \in \Delta$ for which

$$2.11 \quad |\zeta(f)| \leq \max |f(K)|.$$

Let the closed r -neighborhood W still lie in Ω ; and let K be contained in the open ball B of radius R in C^n . Then $B \cap W = L$ is compact. For $x \in L$ (using the maximality and the theorem [3], p. 627) we can find f in H (abbreviation for $\text{Hol}(\Omega)$) such that

$$2.12 \quad |f(x)| > 1$$

while

$$2.13 \quad \max |f(K)| < 1;$$

and we can find $f_1, \dots, f_p \in H$ such that 2.13 holds for all of them, while for $x \in L$, 2.12 holds for at least one of them. Make

$f_{p+i} = Z_i/R$, and consider the "polyhedral domain"

$$P = \{|f_1| \leq 1\} \cap \dots \cap \{|f_{p+n}| \leq 1\}.$$

Then $K \subset P$. Now let $\zeta \in \Delta_K$. Let $\lambda_i = \zeta(f_i)$. If the functions $f_i - \lambda_i$ have a common zero on P , then this point "represents" ζ , because every function in H can be uniformly approximated by polynomials in f_1, \dots on P (Oka-Weil theorem). In the contrary case one can, at each point λ of P , find g_1, \dots, g_{p+n} holomorphic

in a polycylinder, such that

$$2.14 \quad \sum (f_i - \lambda_i) g_i = 1.$$

(This is because 2.1 is true when Ω is a polycylinder - an elementary fact.) Then by [2], theorem 4 bis, we can solve 2.14 with g_i holomorphic in a neighborhood of P . Replacing these by polynomials in f_1, \dots we can make

$$\max |b(K)| < 1$$

where $b = \sum (f_i - \lambda_i) b_i - 1$, $b_i \in H$. But $\zeta(b) = -1$. This contradiction of 2.11 proves 2.1.³⁾

Our original idea was to study σ -Banach algebras by means of Banach algebras; but in algebras of holomorphic functions it now seems proper to work the other way around.

For any set K in C^n we let $\text{Hol}(K) = H$ be the class of

³⁾ Actually, a few pages later Cartan proves a result ([2], p. 63, Cor.) which can be used more neatly than "4 bis". We proceeded as we did in order that we might point out that Cartan's global result (the Corollary) is a result of his local proposition ("4 bis") plus the purely intrinsic approximation technique involved in 1.11.

functions holomorphic in a neighborhood of K , subject to the usual identification (see [2]). If K is compact, we can norm H :

$$2.15 \quad ||f|| = \max |f(K)|$$

and if we like, complete it. Every continuous homomorphism satisfies 2.11, and their totality we denote by Δ_K again.

2.2 THEOREM: $K \subset \Delta_K$ and in fact

$$\Delta_K = \bigcap_{f \in H} \text{dom}(f)$$

where $\text{dom}(f)$ is the largest Bereich to which f can be analytically continued.⁴⁾

Let $f_1, \dots, f_p \in H$ and make $f_{p+i} = z_i$ as before. Let H_1 be the algebra generated by f_1, f_2, \dots regarded as functions on $\Omega = \text{dom}(f_1) \cap \dots \cap \text{dom}(z_n)$. Ω is maximal for H_1 . Hence ζ is

4) To obtain $\Delta_K \subset C^n$, an additional condition must be imposed. It suffices to assume that H has generators with schlicht domains of holomorphy. On the other hand, if (as I have been told at this conference) Cartan's theorem extends to non-schlicht domains of holomorphy, then the theorem extends also, except that the intersection has to be understood as an inverse limit.

represented by a point λ of C^n , unique because $\zeta(z_i) = \lambda_i$ determines λ . This essentially proves 2.2.

This algebra is not always finitely generated, so far as I know (except in an extended sense: z_1, \dots, z_n could be called "holomorphic generators"). Yet $\Delta = \sigma(z_1, \dots, z_n)$.

Let $\text{Rat}(K)$ be the algebra of those rational functions whose denominators do not vanish on K . Norm as before. Let Poly be the ring of polynomials. Suppose A is a subalgebra of $\text{Rat}(K)$, and $A \supset \text{Poly}$. In particular, A might reduce to Poly.

2.3 THEOREM:

$$\Delta_K = \bigcup_{f \in A, \|f\| \leq 1} \{|f| \leq 1\}.$$

Proof: Of course, every point on the right side yields a $\zeta \in \Delta_K$. Conversely, given $\zeta \in \Delta_K$ we can find the corresponding point, $\lambda_i = \zeta(z_i)$, and for $f \in A$ we surely have $f(\lambda) = \zeta(f)$.

What becomes of this elementary proposition if the condition Poly $\subset A$ is dropped?

3. SHILOV BOUNDARIES

We have already (before 2.15) given the definition of $H(K)$ for (in particular) compact subsets of C^n . That minimum, closed subset bK of K on which all the functions in $H(K)$ attain their maximum modulus (on K) may be called the Shilov boundary of K . One could replace K by Δ_K (see 2.2) and arrive at an isometric algebra; and then one could consider $b\Delta_K$. From the minimum property of bK , it follows that $b\Delta_K$ lies in K and thus coincides with bK .

Now let A be a commutative Banach algebra with unit; let a_1, \dots, a_n be elements thereof, and let $K = \sigma(a_1, \dots, a_n)$ as defined in §1. There is then a mapping

$$3.1 \quad \Delta \longrightarrow K$$

or even

$$3.11 \quad \Delta \longrightarrow \Delta_K$$

(where Δ is the space of continuous non-zero complex-valued homomorphisms of A). If this were implemented by a continuous homomorphism

$$3.2 \quad H(K) \longrightarrow A$$

then we could at once assert that in 3.1 or 3.11, the Shilov boundary Γ of Δ is mapped on a set including bK , and this is in fact true. By the main theorem of [1], functions in $H(K)$ do give rise to elements of A , and when A is semi-simple, to unique elements. But the isomorphism thus set up on $H(K)$ (which is incomplete) is not usually continuous, except when sup norms are used. This, however, is all that is needed for the result

$$3.3 \quad \Gamma \longrightarrow \Gamma_1 \supset bK ;$$

and in any case a simple direct argument yields 3.3 even without semi-simplicity.

Therefore, points of bK are permanent points of the joint spectrum. On the other hand,

3.4 if $(\lambda_1, \dots, \lambda_n)$ belongs to the joint spectrum and there is an element $a \in A$ such that $\zeta(a) = 0$ when $\zeta(a_i) = \lambda_i$ ($i = 1, \dots, n$) and a is not a topological zero divisor, then $(\lambda_1, \dots, \lambda_n)$ is not a permanent point.

The reason is that (by a construction to be published soon)

there is an extension algebra B of A in which the element a has an inverse. Consequently those $\zeta \in \Delta_A$ for which $\zeta(a_i) = \lambda_i$ are not extendable to B .

It would be good to know the converse, that is: if $\zeta(a) = 0$ whenever $\zeta(a_i) = \lambda_i$ implies that a is a topological zero divisor, then $(\lambda_1, \dots, \lambda_n)$ is a permanent point. It would follow that finitely many non-permanent homomorphisms (or, in a sense, maximal ideals) could all be removed by a single extension.

Combining 3.4 with the preceding result we have

3.5 If $(\lambda_1, \dots, \lambda_n) \in bK$ then each $a_i - \lambda_i$ is a topological zero divisor.

3.6 THEOREM: Let a_1, \dots, a_n be elements of a commutative σ -Banach algebra with 1. Let $(\lambda_1, \dots, \lambda_n)$ belong to the Shilov boundary of $\sigma_n(a_1, \dots, a_n)$ for almost all n . Then each $a_i - \lambda_i$ is a topological zero divisor.

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ALGEBRAIC PROPERTIES OF
CLASSES OF ANALYTIC FUNCTIONS

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1. INTRODUCTION

Many interesting function spaces and algebras can be obtained by completing a space of polynomials in an appropriate topology. This paper deals with the algebra B of bounded analytic functions in the open unit disc. We shall characterize the dual space of strictly continuous linear functionals on B , showing that they may be regarded as a space of functions, analytic in the open disc, whose Hadamard product with each bounded function is continuous in the closed disc. Some information is then obtained about the strictly closed ideals of the algebra B .

2. TERMINOLOGY

Let D be the open unit disc, $|z| < 1$. By A , B , and C , we mean (respectively) the algebra of all functions analytic in D , all those that are bounded in D , and all those that are uniformly continuous in D . The algebra C is thus the algebra of functions analytic in D , continuous in \overline{D} . Clearly, $A \supset B \supset C$. Let P be the subalgebra of complex polynomials, A and C can be obtained as completions of P . For A , we choose the compact-open

topology k of uniform convergence on each compact subset of D ;

for C , we choose the topology σ of uniform convergence on D .

(k is metrizable, and σ is normable; one may use the norm

$$\|f\| = \sup_{z \in D} |f(z)|.)$$

Information about the structure of A is fairly complete:

(see [4], [5], [7])

(i) The polynomials are k -dense in A .

(ii) Every principal ideal is k -closed.

(iii) The closed maximal ideals of A are the fixed

ideals M_a associated with points a of D ; $M_a = \{f \in A \mid f(a) = 0\}$.

(iv) The closed primary ideals are powers of the M_a .

(v) Every closed ideal in A is principal, and is the intersection of primary ideals.

(vi) Every closed prime ideal is maximal.

There are, of course, non-closed maximal ideals, and non-maximal prime ideals; however, the structure of the closed ideals of A is quite clear.

Due to the work of Wermer, Beurling, Arens, Rudin, and others, the structure of the algebra C has also become known (see [6]). However, the larger algebra B is less well understood;

important contributions have been made by Kakutani, Chevalley, Carleson, and Rudin. It has been customary to study B as a Banach algebra, extending to it the norm topology σ on C . This has certain unwelcome consequences:

(i) The polynomials are no longer dense in B .

(In fact, C is a proper closed subalgebra.)

(ii) As a Banach space, B is not separable.

(A. E. Taylor)

(iii) Principal ideals are not always closed.

Proof: Let $p_n(z) = 1 + z + \dots + z^n$, $q_n = (p_0 + p_1 + \dots + p_n)/(n+1)$ and $f_n(z) = \sqrt{1-z} q_n(z)$. Then, $(1-z)f_n(z) \rightarrow \sqrt{1-z}$ uniformly in D , but $\sqrt{1-z}$ does not belong to the ideal generated by $1-z$.

(iv) Closed ideals are not always principal.

Proof: Let $I = \{f \in B \mid \lim_{x \rightarrow 1} f(x) = 0\}$. This is a σ -closed ideal in B . Suppose $I = (g)$. Since $z-1$ lies in I , g has no zeros in D . Accordingly, \sqrt{g} lies in I ; this implies that $1/\sqrt{g}$ is in B , and I would be all of B .

(v) There are closed maximal ideals in B not of the form M_a .

In fact, since B is a normed algebra, all maximal ideals are closed; the closed ideal I above is contained in a maximal ideal M which cannot be a fixed ideal.

For the algebra C , the work of Rudin shows that every closed ideal, while not necessarily principal, is the closure of a principal ideal. This is clear for an ideal M_a with $|a| < 1$, but is not so immediate for $|a| = 1$. In fact, however, the ideal M_1 of functions in C which are zero at $z = 1$ is generated by $z - 1$; any function f in M_1 is the limit, uniformly in \overline{D} , of a sequence $f_n(z)(1 - z)$. This can be seen at once as follows. Let I be the uniform closure in C of the ideal $(z - 1)$, and let L be any continuous linear functional on C which vanishes on I . In particular, then, $L(z^n(1 - z)) = 0$ for $n = 0, 1, \dots$ and there is a constant c such that $L(z^n) = c$ for all n . Hence, $L(f) = cf(1)$ for every $f \in C$, and L must also vanish on M_1 . Since this holds for any choice of L , $I = M_1$.

This fact is no longer true for the algebra B . A maximal ideal of B which is not one of the ideals M_a for $|a| < 1$ is never the closure of a principal ideal. Suppose, for example, that $M = \overline{(g)}$, the closure of the principal ideal generated by g . There must be a

sequence $a_n \in D$ such that $g(a_n) \rightarrow 0$. Let $I(\{a_n\})$ be the closed ideal of all functions f in B such that $\lim f(a_n) = 0$. This contains g , and must therefore be M itself. However, consideration of Blaschke products shows that a sufficiently rarified subsequence $\{\beta_n\}$ of $\{a_n\}$ can be chosen so that $I(\{\beta_n\})$ is a proper closed ideal properly containing M .

For B , in the uniform norm topology, we have therefore the following picture (here $C^*[D]$ denotes the algebra of all continuous functions on D):

$$C \subset B \subset C^*[D]$$

$$\overline{D} \xleftarrow{\theta'} \mathcal{M} \xleftarrow{\theta} \beta D$$

where the second line lists the spaces of maximal ideals. It is easily seen that the mapping θ' is onto, but that it is not one-to-one. The following argument shows that θ is not one-to-one. Choose two sequences $\{a_n\}, \{\beta_n\}$ tending to 1 in D such that $d(a_n, \beta_n)$, the non-Euclidean distance, tends to zero. From Pick's theorem, one sees that $I(\{a_n\}) = I(\{\beta_n\})$. Define two homomorphisms on C^* by $H'(f) = \text{LIM } f(a_n)$, $H''(f) = \text{LIM } f(\beta_n)$, where LIM is a particular Banach limit. It is easily seen that H' and H'' are distinct on C^* , but agree on B . Whether the map θ is

onto remains an open question.

This suggests that it might be more suitable to study B with a topology different from σ . Since B can be regarded as a subalgebra of $C^*[D]$, it is natural to use in B a topology appropriate to the larger algebra; since D is locally compact, a natural choice is the so-called strict topology β (see [1]). Let $C_0[D]$ denote the continuous complex valued functions on D which "vanish at infinity" (i. e. $\lim_{|z| \rightarrow 1} \phi(z) = 0$).

DEFINITION: A net of functions $f_\alpha \in C^*[D]$ is strictly convergent to f (written $f_\alpha \xrightarrow{\beta} f$) if for each $\phi \in C_0[D]$

$$\phi f_\alpha \xrightarrow{\sigma} \phi f.$$

One easily sees that B is closed in $C^*[D]$. (Remark: it is interesting to observe that Carleson [3] was led to consider a similar condition for individual functions ϕ , in his study of B .)

Specialized to B , the strict topology has certain simple properties:

- (i) The topology β is topologically complete, locally convex, but not metrizable.
- (ii) On uniformly bounded sets in B , β is equivalent to pointwise convergence. (Accordingly, bounded sets are sequentially compact.)

(iii) The polynomials are dense in B . (Landau)

(iv) Let $f \in B$, and for any λ , $0 \leq \lambda \leq 1$, let

$f_\lambda(z) = f(\lambda z)$. Then, $f_\lambda \xrightarrow{\beta} f$ as $\lambda \uparrow 1$.

(v) Let $g \in B$ and have no zeros (in D). Then,

$g^\lambda \in B$ for each $\lambda > 0$, and $g^\lambda \xrightarrow{\beta} 1$ as $\lambda \downarrow 0$.

The last two particularly show the contrast between the topologies β and σ . This is also shown by the following:

$a_n z^n \xrightarrow{\sigma} 0$ if and only if $\lim a_n = 0$

$a_n z^n \xrightarrow{\beta} 0$ if and only if $a_n = O(1)$

$a_n z^n \xrightarrow{k} 0$ if and only if $\limsup |a_n|^{1/n} \leq 1$.

3. THE DUAL SPACE OF B .

Any strictly continuous linear functional L on B can be extended to a continuous functional on the larger space $C^*[D]$.

From known results, we may represent L by

$$L(f) = \iint_D f(z) d\mu(z)$$

where μ is a bounded Radon measure on the open disc D (see [2]).

On B , however, it is natural to expect that the dual space may also be represented by a space of analytic functions. In the present case, this is made easier by the observation that D and \overline{D} are both semi-groups under multiplication. Following a procedure used elsewhere [1], let U_w be the linear transformation of B into itself defined by $U_w(f) = f_w$ where $f_w(z) = f(wz)$, and $|w| \leq 1$. Let L be any continuous linear functional on B . Then, we may construct a linear transformation T on B by setting $T(f) = F$ where

$$F(w) = L(U_w f).$$

THEOREM 1. T is a continuous linear transformation from $\langle B, \beta \rangle$ into $\langle C, \sigma \rangle$.

Note that $L(f) = F(1)$. This theorem has a number of analytical consequences. Let $L(z^k) = c_k$; since $z^k \xrightarrow{\beta} 0$, $\lim c_k = 0$. Set $h(w) = \sum c_n w^n$. Then h is analytic in D , and if $f(z) = \sum a_n z^n$ is any function in B ,

$$F(w) = \sum a_n c_n w^n$$

is continuous in $|w| \leq 1$, and

$$L(f) = F(1) = \lim_{r \uparrow 1} \sum a_n c_n r^n.$$

Thus, the dual space of B can be regarded as a subspace of A , such that $f * h$ (the Hadamard product of f and h) lies in C for every $f \in B$.

Specific examples of representing functions h can be found.

THEOREM 2: Any function h in the class H_2 defines a strictly continuous functional on B .

Proof of this uses the fact that a functional on $C^*[D]$ which is β -continuous on the β -bounded sets is in fact continuous.

THEOREM 3: Let $|a_n| < 1$ with $\lim |a_n| = 1$, and let $\sum |b_n| < \infty$. Then, $L(f) = \sum f(a_n) b_n$ is a continuous functional.

The corresponding analytic function is $h(w) = \sum b_n (1 - a_n w)^{-1}$ which need not belong to H_2 .

When h is analytic in the closed disc \overline{D} , the corresponding functional L is k -continuous on B . Such functionals can be shown to be dense in the full dual of B , in the bounded-open topology, i. e. uniform convergence on bounded subsets of B .

4. IDEALS IN B

In a topological algebra in which multiplication is at least continuous in each factor, the closure of an ideal is an ideal.

Hence, a maximal ideal is either closed or dense. When the set of regular elements (units) is open, every maximal ideal is closed.

In B , the set of regular elements is not open.

THEOREM 4. The strictly closed maximal ideals of B are the fixed ideals M_α associated with points $\alpha \in D$.

THEOREM 5. If $g(z)$ is either a polynomial with no zeros on the boundary of D , or a Blaschke product, then the principal ideal generated by g is strictly closed, and is the intersection of closed primary ideals.

It is not true that every principal ideal in B is closed.

THEOREM 6. $1 - z$ generates a proper dense ideal in B .

COROLLARY. For any function $f \in B$, $(1-z)f(z)$ generates a non-closed principal ideal in B .

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In A , any function g with no zeros in D is a unit, and thus generates A . In B , g is a unit if $1/g$ is bounded in D . What can be said about (g) if merely $g(z) \neq 0$ in D ? There is evidence both for and against the following

CONJECTURE: If $g \in B$, and g has no zeros in D , then (g) is dense in B .

THEOREM 7. If $g \in B$ and the real part of $1/g$ is bounded from below in D , then (g) is dense.

COROLLARY. If g, g^i and g^{-i} are in B , then (g) is dense.

Another special case in which the conjecture can be verified requires a rather interesting behavior for g .

THEOREM 8. Let $g \in B$, and have no zeros in D . Suppose moreover that there are positive constants M and N such that

$$\left| \frac{g(z)^N}{g(\lambda z)} \right| \leq M$$

for all $|z| < 1$ and $0 \leq \lambda \leq 1$. Then, (g) is dense in B .

COROLLARY. If g is analytic in \overline{D} , and has no zeros in D , then (g) is dense in B .

It is clear that a proof of the conjecture would reduce the study of strictly closed ideals in B to a consideration of the ideal generated by two Blaschke products with no common zeros; for, if I is a closed ideal in B , and $f \in I$, then $f = bg$ where g has no zeros in D ; if (g) is dense, then $b \in I$. The argument used by Schilling [7] can then be applied.

The substance of the conjecture can be given different forms.

THEOREM 9. Let g have no zeros in D . Then, (g) is dense in B if and only if the quotient algebra $B/(\overline{g})$ obeys the ascending chain condition for closed ideals.

THEOREM 10. The conjecture is equivalent to the assertion that g lies in the closure of (g^2) , for every non-vanishing g .

In the opposite direction, however, we have the following result; it shows that a proof of the conjecture must make full use of the non-metric nature of the strict topology.

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THEOREM 11. Let $g(z) = \exp (z+1)(z-1)^{-1}$. Then, there does not exist a bounded net $\{f_\alpha\}$ with $f_\alpha g \xrightarrow{\beta} 1$.

5. FURTHER CONSIDERATIONS

As indicated in Section 2, the strict dual space of B is also a space of analytic functions. It is not difficult to see that it is also an algebra, under the operations of addition and Hadamard multiplication. It would therefore be of interest to know more of its structure. The full endomorphism algebra $\Sigma(B)$ is also of interest; the differentiation operator D is not β -continuous, but its role can be filled by the difference operator

$$\Delta(f)(z) = \frac{f(z) - f(0)}{z}.$$

It is clear that some of the discussion given above will hold also for the algebra $B(D)$ where D is a multiply connected region in the plane (at least when D has finite connectivity). One may also consider the space $B(D; E)$ of bounded vector-valued analytic functions. This is no longer an algebra, but is a module over the algebra $B(D)$. One would then seek to describe the strictly closed submodules of $B(D; E)$. Some results of this nature have been

obtained [2] for the full module of bounded continuous E -valued functions on D : If M is a submodule of $C^*[D:E]$ such that $M(p) = E$ for each $p \in D$, then M is strictly dense in $C^*[D:E]$.

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1. INTRODUCTION. An outstanding problem in the theory of Banach algebras R , here always assumed commutative, with unit but without radical, is the characterization of the closed ideals I and in particular the relationship between these ideals and the set of maximal ideals. In order to make methods of analysis available, the natural approach to the problem is to study functionals $F(x)$ taking real or complex values and, e. g., vanishing on I . In 1 - 1 correspondence with I , there is a subadditive (s. a.) and submultiplicative (s. m.), homogeneous and continuous functional $F(x)$ defined by
$$F(x) = \inf_{i \in I} \|x - i\| = \|x\|_I.$$
 On account of the trivial relationship between F and I , these functionals are of no interest in the present context; most existing results have been obtained with the aid of the linear functionals, existing on any Banach space, which have the very serious drawback of having no connection with the multiplicative structure of the space. The only exceptions are the maximal ideals for which the Gelfand functionals, here denoted by $f_M(x)$, exist and reproduce multiplication.

A very natural question in this situation is this: if we restrict or modify the conditions (s. a.) and (s. m.) above without

going as far as Gelfand, do we in this way get any new tools for the study of closed ideals? In considering this question it has turned out that the concept of a multiplicative functional is of some interest while other obvious modifications are equivalent to the Gelfand functionals. Since the statement and proof of the latter negative result is so simple, we shall give it here. As to the theory of multiplicative functionals, we shall here discuss only those problems that are of interest from a function-theoretic point of view. No complete theory is presented; our principal aim is to point out the interesting problems that are connected with this concept.

THEOREM. Let $F(x)$ ($\neq 0$) be a continuous functional, $F(0) = 0$. (a) If $F(x)$ is multiplicative and subadditive, taking real values, then $F(x) = |f_M(x)|^a$ for some $a > 0$ and some M . (b) If $F(x)$ is linear, homogeneous and submultiplicative, then $F(x) = c \cdot f_M(x)$ for some M and some c , $|c| \geq 1$.

Proof. (a) $F(x) = 0$ on a closed ideal I . If $X \in R/I$, $G(X) = F(x+i)$, $x \in X$, is uniquely defined and vanishes when, and only when, $||X||$ does. Since $G(X)$ is subadditive, it is equivalent to $||X||$, and since G is multiplicative, R/I is, by a

known theorem, isomorphic to the complex numbers. Thus $I = M$ for some M and since $F(x) = F(\lambda e)$, $x = \lambda e \in M$, the result follows.

(b) Obviously $F(e) = c$, $|c| \geq 1$. Since

$$|F((\lambda x + y)^2)| = |\lambda^2 F(x^2) + 2\lambda F(xy) + F(y^2)| \leq |\lambda F(x) + F(y)|^2$$

we find, choosing $x = e$ and $\lambda = -F(y)c^{-1}$, $F(y)^2 = cF(y^2)$. If $F(x) = 0$, the relation $c \cdot F(xy) = F(x)F(y)$ is trivial; if $F(x) \neq 0$, we choose in the relation above $\lambda = -F(y)F(x)^{-1}$, and find $c \cdot F(xy) = F(x)F(y)$.

2. DEFINITION. $w(x)$ is a multiplicative functional if $w(x)$ is defined on R taking real values and satisfies the conditions

$$(1) \quad w(xy) = w(x) \cdot w(y)$$

$$(2) \quad \lim_{||x|| \rightarrow 0} w(x) = 0, \quad \lim_{x \rightarrow e} w(x) = 1.$$

If $w(x)$ is bounded for $||x||$ bounded, we use the term bounded multiplicative functional. Note that if $w(x)$ is only defined on the group G of elements having an inverse, it can be extended

to R by putting $w(x) = 0$, $x \notin G$.

An obvious consequence of the definition is that $w(x)$ is continuous on G . Namely, if $x_n \rightarrow x_0$ and x_0^{-1} exists, then $x_n x_0^{-1} \rightarrow e$ and

$$\lim_{n \rightarrow \infty} w(x_n) w(x_0)^{-1} = \lim_{n \rightarrow \infty} w(x_n x_0^{-1}) = 1$$

by (2). Another consequence of (2) is that $w(\lambda e) = |\lambda|^a$, $a > 0$.

Let λ_0 be a point outside the spectrum σ_x of x , so that $y = x - \lambda_0 e$ has an inverse. If $u_x(\lambda) = u(\lambda) = \log w(y - \lambda e)$, we find, if $|\lambda - \lambda_0| \leq \rho$ does not meet σ_x ,

$$\begin{aligned} \frac{1}{n} \sum_{v=0}^{n-1} u(\lambda_0 + \rho e^{2\pi i \frac{v}{n}}) &= \frac{1}{n} \sum_{v=0}^{n-1} \log w(y - \rho e^{2\pi i \frac{v}{n}}) \\ &= \frac{1}{n} \log w(y^n - \rho^n) \\ &= \log w(y) + \frac{1}{n} \log w(e - \rho^n y^{-n}). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \left| |y^{-n}|^{\frac{1}{n}} \right| = \text{Max}_{\lambda \in \sigma} |\lambda - \lambda_0|^{-1} < \frac{1}{\rho}$, the right hand

side tends to $u(\lambda_0)$ by (2), as $n \rightarrow \infty$. We find

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\lambda_0 + \rho e^{i\theta}) d\theta = u(\lambda_0)$$

and conclude that $\log w(x - \lambda e)$ is harmonic outside σ_x .

Let $\bar{u}(\lambda)$ be a conjugate function of $u(\lambda)$ and we are interested in the function $f_x(\lambda) = u + i\bar{u}$ in the component of the complement of σ containing the point at infinity. $f_x(\lambda)$ is analytic and $f'_x(\lambda)$ is also single-valued there. Since $u(\lambda) = a \log |\lambda| + o(1)$, $f(\infty) = 0$.

If

$$f'_x(\lambda) = \frac{a}{\lambda} + \frac{L(x)}{\lambda^2} + \dots$$

we shall prove that $L(x)$ is a bounded linear functional on the algebra R .

Since $u_{cx}(\lambda) = u_x(\frac{\lambda}{c}) + a \log |c|$, we have $f'_{cx}(\lambda) = \frac{1}{c} \cdot f'_x(\frac{\lambda}{c})$.

This yields, for ρ sufficiently large,

$$L(cx) = \frac{1}{2\pi i} \int_{|\lambda|=\rho|c|} f'_{cx}(\lambda) \lambda d\lambda = \frac{c}{2\pi i} \int_{|\lambda|=\rho} f'_x(\lambda) \lambda d\lambda = cL(x).$$

Furthermore

$$\log \left\{ \frac{w(x-\lambda e)w(y-\lambda e)}{w(x+y-\lambda e)} \right\} = a \cdot \log |\lambda| + \log w \{ e^{-\lambda^{-1}xy(x+y-\lambda e)^{-1}} \}.$$

It is easy to see that the last term is $O(\lambda^{-2})$, $|\lambda| \rightarrow \infty$, which means that $f'_x + f'_y - f'_{x+y}$ has a vanishing second order term in its Laurent development. We have thus proved that $L(x)$ is a linear

functional. That $L(x)$ is bounded for $||x|| \leq 1$ is a consequence of the fact that $u_x(\lambda)$ is uniformly bounded for, e.g., $|\lambda| \geq 2$.

Conversely, if L is any linear functional, the relation $\log w(x) = \operatorname{Re} \{L(\log x)\}$ defines a multiplicative functional on the connected part G_0 of G , containing the identity. If $G_0 = G$, the multiplicative structure of R thus determines the additive structure; an interesting problem is to decide to what extent this holds for general rings, i.e. to characterize those linear functionals that may be obtained in the way described above.

3. In the following we shall be concerned with bounded multiplicative functionals. Since $w(x)^n = w(x^n) \leq \operatorname{Const.} ||x^n||$, we have

$$w(x) \leq ||x||_{\text{un}} = \sup_M |f_M(x)|.$$

If μ is a non-negative Radon measure on the space of maximal ideals, the relation

$$w(x) = \exp \left\{ \int_M \log |f_M(x)| d\mu(M) \right\}$$

defines a bounded m.f.. The main problem in this section is to

prove a preliminary converse of this result.

THEOREM. Let $\rho_n(a) > 0$ be a continuous function for all complex a and let $\rho_n(a) \rightarrow 0$, $n \rightarrow \infty$, uniformly. If the circles $|z - a| = \rho_n$ intersect the complement Ω of σ_x in sets of capacity ρ_n , then

$$u_x(\lambda) = \int_{\sigma} \log |z - \lambda| d\mu_x(z), \quad \mu_x \geq 0, \quad \lambda \in \Omega.$$

It is obvious that our assumptions imply that σ has no interior points. For $\lambda \in \sigma$ we redefine $u_x(\lambda)$ by the relation

$$u(\lambda) = \overline{\lim_{\substack{\mu \rightarrow \lambda \\ \mu \in \Omega}} u_x(\mu)}.$$

In this way $u(\lambda)$ becomes an upper semi-continuous function. For an arbitrary point $\lambda_0 \in \Omega$ we consider the associated circles

$|z - \lambda_0| = \rho_n$; to simplify the notations we put $\lambda_0 = 0$. By assumption,

there is a sequence of polynomials $P_\nu(z) = z^\nu + \dots$ with

zeros on $|z| = \rho_n$ such that

$$\lim_{\nu \rightarrow \infty} \inf_{\sigma} |P_\nu(z)|^{\frac{1}{\nu}} = \rho_n.$$

We consider the elements $P_\nu(x)$ of R and find, if λ_k , $k = 0, 1, \dots, \nu-1$, denote the zeros of $P_\nu(z)$,

$$\frac{1}{\nu} \sum_0^{\nu-1} u(\lambda_k) = \frac{1}{\nu} \log P_\nu(x) = \log w(x) + \frac{1}{\nu} \log w(x)^{-\nu} P_\nu(x).$$

Since $\{\lambda_k\}_0^{\nu-1}$ is asymptotically equally distributed on $|z| = \rho_n$ and u is bounded above, we find

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\rho_n e^{i\theta}) d\theta \geq u(0) + \lim_{\nu \rightarrow \infty} \frac{1}{\nu} \inf_{\sigma} \log \left| \frac{P_\nu(z)}{z^\nu} \right|.$$

An application of the maximum principle shows that

$$u(\lambda_0) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\lambda_0 + \rho_n e^{i\theta}) d\theta, \quad \lambda_0 \in \Omega.$$

It is now easy to see that we are able to use the general definition of subharmonic functions (based on a sequence of circles) to conclude that $u(\lambda)$ is subharmonic in the whole complex plane. Since $u(\lambda)$ is harmonic on Ω and $u(\lambda) = a \log |\lambda| + o(1)$, $|\lambda| \rightarrow \infty$, the Riesz representation theorem shows that $u(\lambda)$ is a potential of a positive mass distribution on σ .

The above metrical condition becomes more interesting if we observe that the representation formula can be proved by ordinary extension methods (cf. §5 below) if the set of rational

functions is dense among the continuous functions on σ . It is very plausible that there is an intimate connection between multiplicative functionals and approximation by rational functions and that we have here a new way to approach this difficult approximation problem.

4. Even if we know about every separate element x of R that $\log w(x - \lambda e)$ can be represented by a potential, it is a non-trivial problem to decide if all different set-functions μ_x are projections of one and the same set-function μ on M . This is in fact true in a very general case.

THEOREM. There is a positive measure μ on M such that

$$(*) \quad w(x) = \exp \int_M \log |f_M(x)| d\mu(M), \quad x \in G$$

holds for all x such that, e.g., $m\sigma_x = 0$.

What we need to know about σ_x is that the set of rational functions is dense among the continuous functions. This condition is probably best possible.

We shall here outline the proof in the case of two elements x and y ; this case contains the essential features of the problem. It can then be condensed as follows. Given a two-dimensional set S with projections S_x and S_y on the coordinate axis, and given μ_x and μ_y on S_x and S_y respectively, under what conditions are μ_x and μ_y projections of a μ on S ? This problem is equivalent to the marriage problem (in its finite form) and we shall formulate the result for our particular situation.

LEMMA. Let E_x be an arbitrary closed subset of σ_x and define E_y by the relation

$$E_y = \{f_M(y) \mid f_M(x) \in E_x\}.$$

Then E_y is closed and μ_x and μ_y are projections of one and the same μ on \mathcal{M} if and only if, for every choice of E_x ,

$$\mu_x(E_x) \leq \mu_y(E_y).$$

We now construct a decreasing sequence (in absolute value) of rational functions $X_v(\lambda)$ such that $X_v(\lambda) \rightarrow e$, $\lambda \in E_x$, $X_v(\lambda) \rightarrow 1$, $\lambda \in \sigma_x - E_x$. This is possible since any continuous

function on σ_x can be uniformly approximated by such functions.

Similarly, we construct $Y_\nu(\lambda)$ corresponding to σ_y ; here we also

require that $|Y_\nu(\eta)| \geq |X_\nu(\xi)|$, if there is an M such that

$f_M(x) = \xi$ and $f_M(y) = \eta$. It is easy to see that this is possible.

If now $a_\nu = X_\nu(x) \cdot Y_\nu(y)^{-1}$, then $w(a_\nu) \leq 1$, since for all M ,

$|f_M(a_\nu)| = |f_M(X_\nu) \cdot f_M(Y_\nu)^{-1}| = |X_\nu(\xi)| \cdot |Y_\nu(\eta)|^{-1} \leq 1$. We need

only observe, finally, that $\log w(X_\nu) \rightarrow \mu_x(E_x)$ and

$\log w(Y_\nu) \rightarrow \mu_y(E_y)$; the theorem is thus proved.

Simple examples show that for the validity of the above representation formula it is necessary to require at least that σ has no interior points.

5. We shall in the last two sections assume that R has an involution $x \rightarrow x^*$. The above representation formula can then be proved very simply.

THEOREM. For rings with involution the representation formula (*) holds generally for $x \in G$.

If (*) holds for all x , we call $w(x)$ normalized.

It is obvious that $w(x) = w(x^*)$, $x \in G$. Hence, if $y = xx^*$,

σ_y is situated on the real axis and thus (*) holds for y . Since $w(y) = w(x)^2$, (*) holds also for x .

A direct proof is obtained if we observe that $f_M(x)$ is dense among all continuous functions on \mathfrak{M} . Thus $\log |f_M(x)| = \frac{1}{2} \log(f_M(y))$ is dense among the real continuous functions and by considering the elements y^a , $-\infty < a < \infty$, it is easy to see that $\log w(x)$ generates a real bounded linear functional on the dense subset. If $|f_M(x)| \leq 1$, $M \in \mathfrak{M}$, $\log w(x) \leq 0$, and it is then well-known that

$$\log w(x) = \int_{\mathfrak{M}} \log |f_M(x)| d\mu(M)$$

for a certain non-negative set function μ .

6. In closing, we shall set down the ideal problem that is in a natural way associated with the concepts introduced above. We here assume that R has an involution.

Let I be an arbitrary closed ideal. With I are associated the following ideals:

I_0 : the intersection of all maximal ideals $M \supset I$.

I_1 : $x_0 \in I_1$ if any normalized $w(x)$, which vanishes on I , also vanishes for $x = x_0$.

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I_2 : $x_0 \in I_2$, if there exists an $x \in I$ such that

$$\log |f_M(x)| \leq \text{const.} \log |f_M(x)|.$$

I_3 : $x_0 \in I_3$, if for some n , $x_0^n \in I$.

If we use the representation formula we see that I_1 is actually an ideal, not necessarily closed. We have the inclusions

$$I \subset I_3 \subset I_2 \subset I_1 \subset I_0$$

and, for a deeper understanding of the ideal structure of the given algebra R , it is of great interest to analyze the above inclusions.

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LINEAR INEQUALITIES AND CLOSURE PROPERTIES

IN NORMED LINEAR SPACES

Ky Fan

1. In this abstract we shall summarize some results on linear inequalities and closure properties in normed linear spaces. The closure properties discussed here are suggested by certain uniqueness theorems for analytic functions. The study of these closure properties is closely related to existence theorems for linear inequalities.

As usual, the conjugate space of a normed linear space X will be denoted by X^* . The adjoint transformation of a continuous linear transformation A in X will be denoted by A^* . The polar convex set of a convex set C in X will be denoted by \hat{C} , which is defined by

$$\hat{C} = \{ \phi \in X^* \mid \phi(f) \leq 1 \text{ for all } f \in C \}.$$

A convex set C in X is said to be Euclidean-closed, if, for any continuous linear transformation T from X onto any finite dimensional Euclidean space E^n , the image $T(C)$ of C is closed in E^n . Clearly, every weakly compact convex set in X is Euclidean-closed. Also every linear subspace of X (not necessarily closed in X) is Euclidean-closed. On the other hand, even in a Euclidean space,

a closed convex set need not be Euclidean-closed.

THEOREM 1. Let A be a completely continuous linear transformation in a real normed linear space X , let C be a Euclidean-closed convex set in X such that $0 \in C$. Let $g_0 \in X$ and $\lambda \neq 0$ be a real number. Then there exists an $f \in X$ satisfying the relation

$$(1) \quad Af - \lambda f - g_0 \in C,$$

if and only if $1 + \phi(g_0) \geq 0$ holds for every $\phi \in \hat{C}$ satisfying $A^* \phi - \lambda \phi = 0$.

If C is closed in X instead of being Euclidean-closed, the conclusion of Theorem 1 need not hold even for a finite dimensional space X .

When C is a convex cone, relation (1) is a natural generalization of linear inequalities. When C consists of the null-element 0 alone, Theorem 1 becomes Riesz-Schauder's generalization of an alternative theorem of Fredholm (see [11], [12]). Riesz-Schauder's theory has been extended by Leray [7] to locally convex topological vector spaces. Our Theorem 1 can also be extended to such spaces.

For more explicit systems of linear inequalities, we have

THEOREM 2. Let $\{f_\nu\}_{\nu \in I}$ be a family of elements, not all 0, in a real normed linear space X , and let $\{a_\nu\}_{\nu \in I}$ be a corresponding family of real numbers. Let

$$\sigma = \sup \sum_{i=1}^n \lambda_i a_{\nu_i},$$

when $n = 1, 2, 3, \dots$; $\nu_i \in I$ and λ_i vary under the conditions

$$\lambda_i > 0 \quad (1 \leq i \leq n); \quad \left\| \sum_{i=1}^n \lambda_i f_{\nu_i} \right\| = 1.$$

Then there exists a $\phi \in X^*$ satisfying the system of linear inequalities

$$(2) \quad \phi(f_\nu) \geq a_\nu \quad (\nu \in I),$$

if and only if σ is finite. Moreover, if the system (2) has solutions $\phi \in X^*$, but the zero-functional is not a solution, then σ is equal to the minimum of the norms of all solutions of (2).

The proof of Theorem 1 is given in [6], that of Theorem 2 is in [5].

2. We turn now to the closure properties in normed linear spaces. The results summarized below are contained in a joint paper by Davis and myself [4].

Usually a sequence $\{f_n\}$ of elements in a normed linear space X is said to be complete (also "closed" in the literature), if $\phi = 0$ is the only $\phi \in X^*$ satisfying $\phi(f_n) = 0$ ($n = 1, 2, 3, \dots$). There is a vast literature (see, e. g. Levinson [8]) on particular complete sequences in various concrete function spaces. In many cases, the completeness of a sequence in a special function space is a consequence of certain uniqueness theorems for entire functions of exponential type, or more generally, for functions regular and of exponential type in a half-plane (cf. Boas [2; p. 234-237]). A typical uniqueness theorem for such functions states roughly that a function with certain growth properties must vanish identically if it has, in a certain sense, too many zeros (see [2; Chap. 9]). There are more general uniqueness theorems which state that a sequence of suitably separated points at which the function takes very small values has the same effect as a sequence of zeros. Of this type is, for instance, a uniqueness theorem of Cartwright [3] (see also [2; p. 166]). This second type of uniqueness theorem leads us to the following definition of closure property in an abstract (real or

complex) normed linear space X .

Given a sequence $\{a_n\}$ of non-negative numbers, a sequence $\{f_n\}$ of elements of X is said to be $\{a_n\}$ -complete, if $\phi = 0$ is the only $\phi \in X^*$ satisfying all inequalities $|\phi(f_n)| \leq a_n$ ($n = 1, 2, 3, \dots$).

To form $\{a_n\}$ -complete sequences from a given complete sequence, we have

THEOREM 3. Let $\{g_n\}$ be a complete sequence in a Banach space X such that

$$\lim_{n \rightarrow \infty} \|g_n\|^{\frac{1}{n}} = \sigma < \infty.$$

Let $\{z_n\}$ be a sequence of numbers (real or complex according to whether X is real or complex) such that

$$0 < |z_n| < \sigma^{-1} \quad (n = 1, 2, 3, \dots); \quad \lim_{n \rightarrow \infty} z_n = 0;$$

and let

$$f_n = \sum_{k=1}^{\infty} z_n^k g_k \quad (n = 1, 2, 3, \dots).$$

Then for any sequence $\{a_n\}$ of non-negative numbers satisfying $a_n^{1/n} = O(|z_n|)$, the sequence $\{f_n\}$ is $\{a_n\}$ -complete.

As special cases of Theorem 3, we mention

PROPOSITION 1. Let the coefficients of a power series $F(z) = \sum_{k=0}^{\infty} c_k z^k$ be all real, and let $r > 0$ be its radius of convergence. Assume that $\sum_{n=1}^{\infty} \frac{1}{k_n} = \infty$, where $k_1 < k_2 < \dots < k_n < \dots$ is the sequence of all those integers $k \geq 1$ for which $c_k \neq 0$. Then for any two real sequences $\{t_n\}$, $\{a_n\}$ such that $0 < |t_n| < r$, $\lim_{n \rightarrow \infty} t_n = 0$, $a_n \geq 0$, $a_n^{1/n} = O(|t_n|)$, the sequence of functions $\{F(t_n x)\}$ is $\{a_n\}$ -complete in the real Lebesgue space $L^p(0,1)$ for every $p \geq 1$. In case $c_0 \neq 0$, this is also true for the space $C(0,1)$ of all real continuous functions.

PROPOSITION 2. For any two real sequences $\{t_n\}$, $\{a_n\}$ such that $t_n \neq 0$, $\lim_{n \rightarrow \infty} t_n = 0$, $a_n \geq 0$, $a_n^{1/n} = O(|t_n|)$, the sequence of functions $\{\exp(-\frac{x^2}{2} + 2t_n x - t_n^2)\}$ is $\{a_n\}$ -complete in the real $L^2(-\infty, +\infty)$.

Also, a generalization due to Boas [1] of Lerch's theorem is a result concerning $\{a_n\}$ -completeness.

Using Theorem 2, we can prove the following result which characterizes the $\{a_n\}$ -completeness by a certain approximation

property.

THEOREM 4. Let $\{a_n\}$ be a sequence of non-negative numbers. A sequence $\{f_n\}$ of elements in a (real or complex) normed linear space X is $\{a_n\}$ -complete, if and only if, for any $g \in X$ and for any $\varepsilon > 0$, there exist a finite number of coefficients c_1, c_2, \dots, c_m (real or complex according to whether X is real or complex) such that

$$\left\| g - \sum_{n=1}^m c_n f_n \right\| < \varepsilon, \quad \sum_{n=1}^m |c_n| a_n < \varepsilon.$$

As application of Theorem 4, we have

PROPOSITION 3. Let f be a function regular in $|z| < 1$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^2 d\theta \leq M \quad \text{for } 0 \leq \rho < 1,$$

where M is independent of ρ . For any $\varepsilon > 0$, and for any given complex sequence $\{z_n\}$ with $0 < |z_n| < 1$,

$\lim_{n \rightarrow \infty} z_n = 0$, there exist a finite number of coefficients c_1, c_2, \dots, c_m such that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) - \sum_{n=1}^m \frac{c_n}{z_n(1-z_n e^{i\theta})} \right|^2 d\theta < \varepsilon^2, \quad \sum_{n=1}^m |c_n| < \varepsilon.$$

Using Theorem 4 and the uniqueness theorem of Cartwright mentioned above, one can prove

PROPOSITION 4. Let a, b be real and $|a| < b$. Let $\{\lambda_n\}$ be a positive increasing sequence of maximum density greater than $\frac{b}{\pi}$ and such that $\lambda_{n+1} - \lambda_n \geq \delta > 0$. Then for any function f in the complex $L(a, b)$ and for any $\varepsilon > 0$, $\eta > 0$, there exist a finite number of coefficients c_1, c_2, \dots, c_m such that

$$\int_a^b \left| f(t) - \sum_{n=1}^m c_n \exp((\eta + it)\lambda_n) \right| dt < \varepsilon, \quad \sum_{n=1}^m |c_n| < \varepsilon.$$

(For maximum density, see Pólya [10].)

The following theorem involves another type of completeness.

THEOREM 5. Let $\{f_n\}$ be a sequence of elements in a normed linear space X , and let $\{a_n\}$ be a sequence of non-negative numbers. If, for $\phi \in X^*$, the inequalities

$|\phi(f_n)| \leq a_n$ ($n = 1, 2, 3, \dots$) imply $||\phi|| < 1$, then for any $g \in X$ with $||g|| = 1$ and for any $\epsilon > 0$, there exist a finite number of coefficients c_1, c_2, \dots, c_m such that

$$||g - \sum_{n=1}^m c_n f_n|| < \epsilon, \quad \sum_{n=1}^m |c_n| a_n < 1.$$

A well-known theorem of Paley-Wiener [9; p. 100-108] concerning ordinary completeness in a Hilbert space has been generalized by several authors. Using a result on linear inequalities [5; Theorem 22], we can prove the following theorem of Paley-Wiener type for $\{a_n\}$ -completeness.

THEOREM 6. Let $\{f_n\}$, $\{g_n\}$ be two sequences of elements in a normed linear space X . Suppose that there exists a number λ , $0 \leq \lambda < 1$, with the property that

$$||\sum_{n=1}^m c_n (f_n - g_n)|| \leq \lambda ||\sum_{n=1}^m c_n f_n||$$

holds for any finite set of coefficients c_n . If, for some sequence $\{a_n\}$ of non-negative numbers, $\{f_n\}$ is $\{a_n\}$ -complete, then $\{g_n\}$ is also $\{a_n\}$ -complete.

For a Hilbert space, we have

THEOREM 7. Let $\{f_n\}$, $\{g_n\}$ be two sequences of elements in a Hilbert space H . Suppose that for any finite set of coefficients c_n , the inequalities

$$\rho_1 \left\| \sum_{n=1}^m c_n f_n \right\| \leq \left\| \sum_{n=1}^m c_n g_n \right\| \leq \rho_2 \left\| \sum_{n=1}^m c_n f_n \right\|$$

$$\left\| \sum_{n=1}^m c_n (f_n - g_n) \right\|^2 \leq \left\| \sum_{n=1}^m c_n f_n \right\|^2 + \left\| \sum_{n=1}^m c_n g_n \right\|^2$$

hold, where $\rho_1 > 0$, $\rho_2 > 0$ are two fixed constants. Then:

- (i) If $\{f_n\}$ is $\{a_n\}$ -complete, so is $\{g_n\}$.
- (ii) If $\{f_n\}$ is complete orthonormal in H , then $\{g_n\}$ admits a biorthonormal sequence $\{h_n\}$ and every element $u \in H$ has the expansions

$$u = \sum_{n=1}^{\infty} (u, h_n) g_n = \sum_{n=1}^{\infty} (u, g_n) h_n.$$

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FUNCTION ALGEBRAS

A. M. Gleason

1. INTRODUCTION

By a function algebra we shall mean a semi-simple complex Banach algebra with the spectral norm. To any Banach algebra we can associate a function algebra by dividing out the radical and completing in the spectral norm. Thus it appears that, although function algebras are only a special class of Banach algebras, a structure theory for function algebras would be a very long step towards a structure theory for general Banach algebras. We cannot hope for such a theory in the foreseeable future, so it is appropriate to study function algebras satisfying various additional conditions.

To begin with we shall restrict ourselves to separable algebras with a unit, and a subalgebra will always be assumed to have the same unit. Let A be a function algebra and let H be the set of all homomorphisms of A into the complex numbers C , that is, the set of all elements of A^* (dual space) which are multiplicative as well as linear. If H is assigned the weak star topology as a subset of A^* , then it is well known that H is compact. Furthermore, the elements of A can be regarded as continuous functions on H , and this identification embeds A isometrically as a subalgebra of $C(H)$, the algebra of all continuous complex-valued functions on H . For

any subset X of H , A can be mapped homomorphically into $C(X)$ by restriction; this map will generally be norm-diminishing, but it is known that there is a least compact subset B of H called the Shilov boundary such that A is isometrically embedded in $C(B)$. In what follows we will use the letters H and B for the set of all homomorphisms and the Shilov boundary without further introduction.

A function algebra A is said to be finitely generated, more specifically n -generated, if there exists a finite set $\{z_1, \dots, z_n\}$ of elements of A which generate a dense subalgebra of A . In this case the mapping

$$h \longrightarrow [h(z_1), \dots, h(z_n)]$$

of H into C^n is a homeomorphism. If we denote the image set by K , then A can be regarded as an algebra of continuous functions on K , and in fact A is the closure in the uniform norm of those functions on K obtained by restricting the polynomial functions of C^n .

If A has one generator, then it is known that K can be any compact set which does not divide the plane, and Mergelyan has shown that A contains every function which is continuous on K

and analytic at all interior points of K . One of the central objectives of the theory of function algebras is an analogue of Mergelyan's theorem; in this paper we will discuss certain facts which may lead toward this objective.

2. THE METRIC TOPOLOGY OF H

The set H has usually been considered with the weak star topology. While this has the advantage that the resulting space is compact, examples indicate the norm topology of H as a subset of A^* is very significantly related to the analytic aspects of the algebra A .

If $h_1, h_2 \in H$, then we have $||h_1 - h_2|| \leq ||h_1|| + ||h_2|| = 2$.

If A is the algebra of all continuous functions, then

$||h_1 - h_2|| = 2$ for every pair of distinct homomorphisms. At the other extreme, suppose A is the algebra of continuous functions on the closed unit disk which are analytic in the interior. The homomorphisms are precisely the point valuations of the functions and we find $||h_1 - h_2|| < 2$ if and only if h_1 and h_2 both correspond to interior points of the disk (assuming $h_1 \neq h_2$). Moreover, it follows easily from Schwarz's lemma that the interior of the disk is embedded homeomorphically in A^* in the metric topology.

For the general case of an algebra with one generator we can regard H as embedded in C ; in this terminology, we find that $||h_1 - h_2|| < 2$ (for distinct h_1, h_2) if and only if h_1 and h_2 are in the same component of the interior of H , and each component of the interior is embedded homeomorphically. (The author's proof of this depends on a topological question which he has not yet fully clarified.) These examples should show the importance of the following:

2.1. CONJECTURE. If A is a function algebra and if, for every two homomorphisms, $||h_1 - h_2|| = 2$, then A is $C(H)$.

Although the author has been unable to prove this, the following theorem leads us to some interesting insights.

2.2. THEOREM: In any function algebra, if

$$||h_1 - h_2|| < 2 \text{ and } ||h_2 - h_3|| < 2, \text{ then } ||h_1 - h_3|| < 2.$$

In other words, the relation $||h_1 - h_2|| < 2$ is transitive. Since it is trivially symmetric and reflexive it is an equivalence relation. We shall call the equivalence classes of this relation parts

of H . Each part of H seems to be in some weak sense an analytic variety.

Let ϕ be a map of the open unit disk D into a Banach space V . There are apparently several notions of analyticity for such a map, for example

1) $\phi(t) = \sum v_i t^i$ where $t \in D$, $v_i \in V$ and the series converges in the norm for all t .

2) $f \cdot \phi$ is an analytic function for all $f \in V^*$.

If V is a conjugate space, say $V = W^*$, then we may demand that

3) $t \rightarrow \phi(t)(w)$ is an analytic function for all $w \in W$.

All of these notions coincide (and the same is true for analytic functions of several variables).

Suppose that ϕ is an analytic map of D into A^* with range in H . It follows immediately from Schwarz's lemma that the range lies in a single part of H . We conjecture a converse of this fact.

2.3. CONJECTURE. A necessary and sufficient condition that h_1 and h_2 be in the same part of H is that h_1 and h_2 can be connected by a finite chain of analytic images of the unit disks.

The sufficiency of this condition follows from theorem 2.2 and the previous remarks. A proof of the other half would have a number of interesting consequences, among them the existence of point derivations at any homomorphism which is in a part of H containing more than one point.

A point derivation at h is a linear map ψ of A into C satisfying the relation $\psi(ab) = \psi(a)h(b) + h(a)\psi(b)$ for any two elements a and b of A . Suppose ϕ is an analytic map of the unit disk into H which is not constant, and say $\phi(0) = h$. Then the map $A \rightarrow f_a$ where $f_a(t) = \phi(t)(a)$ is a homomorphism of A into the algebra of bounded analytic functions on the open disk and the usual point derivation at 0 (i.e. $f \rightarrow f'(0)$) on the latter algebra induces a point derivation on A at h .

Examples such as we discussed above show that the parts of H are frequently analytic varieties, but sometimes the situation can be more complicated. In the space C^2 let $K = \{[w, z]: |z| \leq 1, w = 0 \text{ or } w = \frac{z}{n} \text{ for some integer } n\}$; K is the union of a countable family of ordinary disks which meet at the origin. Let A be the closure of the polynomial algebra on C^2 in the uniform norm computed over K . The map $h \rightarrow [h(w), h(z)]$ (here w and z are the ele-

ments of A corresponding to the coordinate functions of C^2) carries H onto K and it can be checked that the parts of H are the single points for which $|z| = 1$, with all the rest (i. e. $|z| < 1$) in a single part. Furthermore, the mapping is a homeomorphism of each part; hence the one non-trivial part is not an analytic variety in the usual sense although it is a union of varieties. Probably more complicated cases can be found.

Let P be a part of H . The elements of A can be regarded as bounded functions on P . Designate by A_P the set of all functions on P which are pointwise limits of uniformly bounded directed systems of functions from A . The set A_P is a Banach algebra and it seems appropriate to call its members the bounded analytic functions on P . (In case A is the algebra of functions continuous on the closed unit disk and analytic in the interior and P is the interior part, then A_P is exactly the algebra of all bounded analytic functions on the unit disk).

Assuming the validity of the remarks just preceding 2.1, . . . about the parts of an algebra with one generator, Mergelyan's theorem can be stated as follows:

THEOREM OF MERGELYAN: If A is a function algebra with one generator then a necessary and sufficient condition that a function f defined on H should be in A is that

- 1) f is continuous with respect to the weak star topology of H .

and

- 2) for each part P of H , f restricted to P is in A_P .

Since this statement makes sense for any function algebra, it may be that the proper analogue of Mergelyan's theorem is obtained simply by deleting the words "with one generator" and possibly replacing with the words "with a finite number of generators". It is easy to verify this conjecture for the function algebras obtained by norming the polynomial algebra on certain domains such as smooth polycylinders.

3. DIRICHLET ALGEBRAS

A Dirichlet algebra is a function algebra for which the boundary fits smoothly into the space H , in such a way that we can solve the analogue of the Dirichlet problem in harmonic functions. It appears that this class of algebras is of considerable importance and

is amenable to analysis.

Let A be a function algebra which we regard as made up of functions on H . The real parts of these functions and the functions uniformly approximable by such functions will be called harmonic functions on H . Since the harmonic functions achieve their norms on the Shilov boundary B , each real continuous function on B is the restriction of at most one harmonic function. We shall say that A is a Dirichlet algebra if every real continuous function on B is the restriction of a harmonic function on H .

That Dirichlet algebras exhibit less pathology than the general function algebra is seen in the following theorem.

3.1. THEOREM: In a Dirichlet algebra, H is connected in the weak star topology if and only if B is.

If K is a compact subset of C^n and A is the function algebra obtained by completing the ring of polynomial functions on C^n in the uniform norm over K , then we shall say that K is a Dirichlet set if A is a Dirichlet algebra. It is well known that any region in the plane with smooth boundary is a Dirichlet set. The algebras considered by Arens and Singer (Generalized analytic functions, Trans. Amer. Math. Soc., Vol. 81 (1956), pp. 379-393)

are Dirichlet algebras. Other examples can be given which are, in effect, continuous direct sums of these elementary examples.

Any homomorphism h of A has an integral representation in terms of a positive real Radon measure μ on B , obtained simply by extending the linear functional h over all of $C(B)$. If A is a Dirichlet algebra, the measure μ is unique among real measures, for no real measure annihilates A .

3.2. THEOREM: Let h_1 and h_2 be homomorphisms of a Dirichlet algebra A and let μ_1 and μ_2 be the corresponding real measures on B . Then $||h_1 - h_2|| < 2$ if and only if μ_1 and μ_2 are absolutely continuous with respect to one another. Moreover, if the condition is satisfied, then the Radon-Nikodym derivative of μ_1 with respect to μ_2 is bounded.

For each homomorphism h with corresponding measure μ , there is a least compact subset X of B which supports μ ; it may be characterized as the set of all points of B all of whose neighborhoods (weak star topology) have positive μ measure.* Theorem 3.2 implies that the set X is the same for all homomorphisms in the same part of H .

Let h_0 be a homomorphism of the Dirichlet algebra A , and let μ_0 and X_0 be the corresponding measure and its support set. We can introduce a new norm into A by

$$||A||_0 = \sup \{ |h(a)| : h \in X_0 \}.$$

Dividing out the ideal of functions which vanish on X_0 and completing in the new norm we get a function algebra A_0 . Let H_0 and B_0 be the set of homomorphisms for A_0 and the boundary, respectively. Since A_0 contains a continuous homomorphic image of A , every member of H_0 induces a homomorphism of A by composition and this provides us with an embedding of H_0 into H . With this identification B_0 becomes X_0 and H_0 can be characterized as the union of all parts of H for which the support set is contained in X_0 . It follows that A_0 is itself a Dirichlet algebra, and that the metric for H_0 computed in A_0^* is the same as that computed in A^* . We have thus come to the consideration of a Dirichlet algebra such that the entire boundary is the support set for some homomorphism. Such an algebra we shall call primitive, and such a homomorphism, or the part containing it, we shall call primary. The foregoing argument shows that any homomorphism can be made primary by passing to a suitable quotient algebra.

3.3. THEOREM: In a primitive Dirichlet algebra, any function which is real on the boundary is constant.

Let A be a primitive Dirichlet algebra and let f be a real function on B which is the real part of some function a in A . Any two functions in A having the same real part on B differ by a constant by theorem 3.3, so that if h_0 and h_1 are homomorphisms, $\text{Im}(h_1(a) - h_0(a))$ is determined by f .

3.4. THEOREM: If h_1 and h_0 are in the same part of H , then the map $f \rightarrow \text{Im}(h_1(a) - h_0(a))$ defined above is a continuous linear functional and has a unique extension over the space of real continuous functions on B . Moreover this linear functional is represented by a real measure on B having the same null sets as the measure representing h_0 .

Previously we assumed that each real function on B had a harmonic extension over H ; now we see that we can define its harmonic conjugate which is determined up to an additive constant on each part of H . The harmonic conjugate is continuous when H is assigned the norm topology, but of course it may be unbounded.

Consider now a primitive Dirichlet algebra A , a primary

part P and the measure μ_0 associated with one of the homomorphisms in P . We can form the space M of (classes of) bounded μ_0 -measurable functions on B with the essential supremum as norms. Since μ_0 is a Borel measure, continuous functions are μ_0 -measurable, so that $C(B)$ is naturally mapped into M . This map is an isometry since μ_0 assigns positive measure to every open set. Moreover, M is a conjugate space and, because μ_0 is regular, the image of $C(B)$ is weak star dense in M . The algebra A is also embedded in M (via its embedding in $C(B)$) and has a weak star closure A_M . We can extend each element f of A_M over P by the integral representation formula

$$f(h) = \int_B f(b) d\mu(b)$$

where μ is the measure on B corresponding to h . (We know that, for $h \in P$, $d\mu = \phi \cdot d\mu_0$ where ϕ is an L_1 function.) This gives us a mapping of A_M into a class of functions on P and it can be shown that it is a homomorphism of A_M onto A_P (as defined in §2). It seems quite likely that this map is actually one-to-one, but the author cannot prove this. Assuming this is so, then the measurable functions of A_M are the "boundary values" of the

analytic functions of A_P , and a theorem similar to Fatou's theorem for the unit disk seems likely. These conjectures depend largely on showing that P has sufficiently many points, and this, in turn, is closely related to conjecture 2.1.

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1. INTRODUCTION

In this paper, we shall extend the following theorem of Fatou [3; p. 147] from the class of analytic functions in the unit disc to certain classes of generalized analytic functions studied by Arens and Singer [2].

THEOREM 1.1. Let F be a bounded analytic function in the unit disc, $|z| < 1$. Then

$$F(e^{i\theta}) = \lim_{r \rightarrow 1} F(re^{i\theta})$$

exists for almost every θ , and

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) P_r(\theta-t) dt$$

where P_r is Poisson's kernel.

This result will not be established in the full generality of [2], but in the "archimedean-ordered and discrete" case of that paper. We begin by summarizing briefly the results of [2] which will be needed here.

Let G be a subgroup of the additive group of real numbers, considered as a discrete topological group, and let G_+ denote the set of elements in G which are not less than 0. Let Δ be the set of all characters of G_+ , i.e., homomorphisms of G_+ into the unit disc of the complex plane. We make Δ into a topological space, using the topology of uniform convergence on compact (finite) subsets of G_+ . If Γ is the character group of G , each element of Γ determines a homomorphism of G_+ into the disc, and the so determined (one-one) embedding of Γ in Δ is a homeomorphism of Γ with a closed subset of Δ .

In the classical case, when G is the group of integers, the space Δ is (homeomorphic to) the unit disc and Γ is the unit circle.

Each element ζ in Δ is uniquely representable in the form

$$(1.11) \quad \zeta = \rho \alpha$$

where ρ is a non-negative element of Δ and α is in Γ .

The "disc" Δ contains a point ζ_0 , which we shall call the origin of Δ , defined by

$$(1.21) \quad \zeta_0(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0. \end{cases}$$

At times, we shall write simply $\zeta = 0$ to mean $\zeta = \zeta_0$.

If ρ is a positive element of Δ , $0 < \rho(x) \leq 1$, there is a positive real number r , $0 < r \leq 1$, such that

$$(1.31) \quad \rho(x) = r^x.$$

The correspondence determined by (1.31) is a homeomorphism between the real segment of Δ and the unit interval $(0, 1)$.

Consider the Banach algebra $L_1(G)$, the multiplication being

$$(f * g)(x) = \int_G f(x-y)g(y)dy$$

and the norm

$$\|f\|_1 = \int_G |f(x)| dx.$$

Let A_1 be the subalgebra of $L_1(G)$ consisting of those functions f which are supported on G_+ . Let $H(A_1)$ be the space of complex homomorphisms (regular maximal ideals) of the algebra A_1 . Each element h of $H(A_1)$ is uniquely representable in the form

$$(1.41) \quad h(f) = \int_{G_+} f(x)\zeta(x)dx$$

for some ζ in Δ . Conversely, given any ζ in Δ , (1.21) determines an element h in $H(A_1)$. The one-one correspondence determined by (1.21) is a homeomorphism of Δ and $H(A_1)$. We thus identify Δ and $H(A_1)$ and may refer to an element of Δ as a character of G_+ or a homeomorphism of A_1 .

The standard Gelfand theory now tells us that each function f in A_1 determines a continuous function \hat{f} on Δ by

$$(1.51) \quad \hat{f}(\zeta) = \int_{G_+} f(x)\zeta(x)dx.$$

In the classical case, the representing functions \hat{f} are those functions continuous on the disc Δ , analytic in the interior, such that the restriction of \hat{f} to Γ has an absolutely convergent Fourier series.

For the algebra A_1 , Γ is the Silov boundary of the space of maximal ideals Δ . That is, for each f in A_1 , the maximum modulus of \hat{f} is taken on the set Γ ; and Γ is the unique minimal closed set in Δ having this property. Because of this general maximum modulus principle, there is, for each ζ in Δ , a regular Baire measure m_ζ on Γ such that for every f in A_1

$$(1.61) \quad \hat{f}(\zeta) = \int_{\Gamma} \hat{f}(a) m_{\zeta}(da).$$

If $\zeta = \rho a$, we have

$$(1.62) \quad \hat{f}(\zeta) = \int_{\Gamma} \hat{f}(a\beta) m_{\rho}(d\beta).$$

The formula (1.62) generalizes the familiar Poisson integral representation of the classical situation.

If Π is the closed half-plane $\operatorname{Re}(w) \geq 0$, each positive ρ , $\rho \neq 1$, in Δ determines a continuous mapping of Π into Δ .

We map the complex number w into the element ρ^w defined by

$$(1.71) \quad \rho^w(x) = \exp [w \log \rho(x)].$$

The mapping defined in (1.71) maps the imaginary axis of Π into the character group Γ . The representation (1.31) of any ρ in Δ makes it clear that the image of the axis under this mapping is the same subgroup of Γ for each ρ , $0 < \rho < 1$. We shall call this subgroup Ω . As Arens and Singer showed, Ω is a dense subgroup of Γ ; also, unless G is (isomorphic to) the group of integers, Ω has Haar measure zero. It is perhaps well to normalize our notation in Ω , defining Ω as the set of all characters in Γ of the form

$$(1.81) \quad a_v(x) = e^{-ivx}.$$

The "harmonic" measures m_ρ , $\rho > 0$, determined by (1.61), are supported on the subgroup Ω . Indeed, if $0 < \rho < 1$, then for each bounded Baire function ϕ on Γ

$$\int_{\Gamma} \phi(a) m_\rho(da) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\rho^{iv}) (1+v^2)^{-1} dv.$$

Or, using (1.81),

$$(1.91) \quad \int_{\Gamma} \phi(a) m_\rho(da) = \int_{-\infty}^{\infty} \phi(a_v) C_u(v) dv$$

where $u = -\log \rho$ and C_u is the Cauchy density

$$(1.92) \quad C_u(v) = u[\pi(u^2 + v^2)]^{-1}.$$

One useful conclusion from (1.91) is that the measures m_ρ , $0 < \rho < 1$, are mutually absolutely continuous.

We conclude our brief review of a portion of the Arens-Singer results by observing that for each f in the algebra A_1 (and for each ρ , $0 < \rho < 1$) the function $\hat{f}(\rho^{u+iv})$ is holomorphic for u positive, and continuous and bounded on all of the half-plane Π . These functions are in fact almost periodic; however, no use will be made of that fact in this paper.

2. THE HARMONIC MEASURES

The origin ζ_0 of the disc Δ (1.21) has as its corresponding harmonic measure the Haar measure of the character group Γ . In this section, we shall establish a useful relation between Haar measure on Γ and the harmonic measures m_ζ , $\zeta \neq \zeta_0$.

Let x_0 be a non-zero element of G_+ . Let Z be the compact subgroup of Γ consisting of those characters α for which $\alpha(x_0) = 1$.

THEOREM 2.1. The character group Γ is locally isomorphic to the direct product of the unit circle, $|z| = 1$, and the subgroup Z .

Proof. By isomorphism, we of course mean topological isomorphism. Consider the neighborhood W of α_0 of the form

$$W = \{\alpha; \alpha(x_0) \neq -\alpha_0(x_0)\}.$$

Then there is a one-one correspondence between W and the product $U \times Z$, where U is the open subset of the unit circle obtained by deleting that number e^{it_0} such that $\alpha_{t_0}(x_0) = -\alpha_0(x_0)$. The correspondence is $\alpha \mapsto (e^{it}, \beta)$, where t is the unique real number between 0 and $\frac{2\pi}{x_0}$ such that $\alpha(x_0) = \alpha_t(x_0)$, and β is the element

$a a_t^{-1}$ in Z . This is clearly an algebraic isomorphism; we need only verify that the topologies are the same. But, this is a routine verification, which we shall omit.

Note that, according to the proof of (2.1), two direct product neighborhoods will cover Γ . It now follows that, except for a normalization, the Haar measure of Γ is locally the product of the Lebesgue measure on the circle and the Haar measure of the compact group Z [1; theorem 2.3]. In the notation of (2.1), we may then state that for any bounded Baire function ϕ on Γ

$$(2.21) \quad \int_W \phi(a) da = K \int_Z \int_U \phi(a_t \beta) dt d\beta$$

where K is a constant of (measure) normalization.

We now establish our result connecting the harmonic measures on Γ .

THEOREM 2.3. Let S be a Borel set in Γ and let ρ be an element of Δ , $0 < \rho < 1$. If for each a in Γ we have $m_{\rho a}(S) = 0$, then S has Haar measure zero.

Proof. Let k_S be the characteristic function of the set S . Again with the notation of (2.1), we may apply the Fubini theorem to (2.21) to conclude that

$$(2.31) \quad \int_W k_S(a) da = K \int_Z d\beta \int_U k_S(a_t \beta) dt.$$

Hence,

$$(2.32) \quad \int_W k_S(a) da \leq K \max_{\beta} \int_U k_S(a_t \beta) dt = K \max_{\beta} M(S_{\beta})$$

where $S_{\beta} = U \times \{\beta\}$ and M is Lebesgue measure on the circle.

Now

$$(2.33) \quad m_{\rho}(S_{\beta}) = m_{\rho\beta}(S) = 0.$$

By (1.91),

$$(2.34) \quad m_{\rho}(S_{\beta}) = \int_{\Gamma} k_S(a\beta) m_{\rho}(da) = \int_{-\infty}^{\infty} k_S(a_v \beta) C_u(v) dv.$$

As $m_{\rho}(S_{\beta}) = 0$ and C_u is a positive kernel, it is clear that

$$(2.35) \quad M(S_{\beta}) = \int_U k_S(a_v \beta) dv = 0.$$

It follows from the inequality (2.32) that

$$\int_W k_S(a) da = 0,$$

and since two neighborhoods W will cover Γ , the Haar measure of S is zero.

This theorem is of course trivial in the classical case, in

which Haar measure is absolutely continuous with respect to each m_ζ , $|\zeta| < 1$. However, in general we know that the measures m_ζ , $\zeta \neq \zeta_0$, are mutually singular with Haar measure, being supported on translates of the one-parameter subgroup Ω (1.8), which has Haar measure zero.

3. THE FATOU THEOREM

The complex-valued function F on the disc Δ is called analytic in the interior of Δ , if F can be uniformly approximated on compact subsets of $\Delta - \Gamma$ by functions \hat{f} , with f in A_1 . We shall prove the following generalization of the Fatou theorem (1.1).

THEOREM 3.1. Let F be a bounded analytic function in the interior of Δ . Then

$$(3.11) \quad F(a) = \lim_{\rho \rightarrow 1} F(\rho a)$$

exists, except on a set which has m_ζ measure zero for every ζ in the interior of Δ . Furthermore,

$$F(\rho a) = \int_{\Gamma} F(a\beta) m_\rho(d\beta).$$

Proof. Let S be the set of a 's in Γ for which

$$\lim_{\rho \rightarrow 1} F(\rho a)$$

does not exist. First, suppose $\zeta = \rho\gamma$, $0 < \rho < 1$. We wish to show that $m_\zeta(S) = 0$. Clearly we may as well assume (by rotating F by the character γ^{-1}) that $\gamma = 1$, or that $\zeta = \rho$. Since the measures m_ρ , $0 < \rho < 1$, are mutually absolutely continuous, it is sufficient to consider the case in which $\rho = e^{-1}$, i. e., that $\rho(x) = e^{-x}$.

We define a function H on the half-plane $\operatorname{Re} [w] > 0$ by

$$H(w) = F(e^{-w}).$$

Now H is a bounded analytic function in the half-plane. The analyticity of H follows from the fact that for each f in A_1 the function $\hat{f}(e^{-w})$ is analytic in w (see section 1).

As is well known, the Fatou theorem in the classical unit disc immediately implies that if H is a bounded analytic function in the half-plane, then

$$H(iv) = \lim_{u \rightarrow 0} H(u+iv)$$

exists for almost every real number v . In the case at hand,

$$H(u+iv) = F(e^{-u} \cdot e^{-iv}).$$

Now plainly

$$\lim_{u \rightarrow 0} H(u+iv)$$

exists if and only if

$$\lim_{\rho \rightarrow 1} F(\rho e^{-iv})$$

exists. Thus, the latter limit exists for almost every v . If k_S is the characteristic function of the set S , then

$$\int_{\Gamma} k_S(a) m_{(1/e)}(da) = \frac{1}{\pi} \int_{-\infty}^{\infty} k_S(e^{-iv}) (1+v^2)^{-1} dv.$$

But, as we have just observed, $k_S(e^{-iv}) = 0$ for almost every v .

Thus

$$m_{(1/e)}(S) = 0.$$

This completes the proof that for $\zeta = \rho \gamma$, $0 < \rho < 1$,

$$m_{\zeta}(S) = 0.$$

Then, by Theorem 2.3, the Haar measure of S is zero. If the function F is extended to Γ by (3.11), we clearly have

$$F(\rho a) = \int_{\Gamma} F(a\beta) m_{\rho}(d\beta).$$

By arguments similar to those above, one can establish generalizations to this context of other boundary-value theorems concerning analytic functions in the unit disc. As one example, we mention the Riesz theorem [4] on the existence of radial limits for analytic functions in some Hardy class H_p .

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ON RINGS OF BOUNDED ANALYTIC FUNCTIONS

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1. Let D be a non-empty bounded open set in the Gaussian plane. We do not assume that D is connected, but it is assumed that the boundary $Bd(D) = \bar{D} - D$ of D has no isolated point. Let $B(D)$ be the ring of all bounded single-valued analytic functions $f(z)$ defined on D . $B(D)$ is a normed ring with respect to the norm:

$$(1) \quad ||f|| = ||f||_D = \sup_{z \in D} |f(z)|.$$

The properties of this ring were discussed in [3].

The purpose of this paper is to discuss the boundary behavior of a function $f(z)$ from $B(D)$ in terms of the theory of normed rings.

2. We begin with definitions and notations from the theory of normed rings. By a ring we always mean a commutative ring with unit 1 over the field of complex numbers, and by a homomorphism of one ring into another, we always mean a homomorphism which maps the unit into the unit and which admits the scalar multiplication.

A non-negative real-valued function $||f||$ defined on a ring $R = \{f\}$ is called a quasi-norm if it satisfies the following conditions:

$$(i) \quad ||f + g|| \leq ||f|| + ||g||, \quad (ii) \quad ||fg|| \leq ||f|| \cdot ||g||, \quad (iii) \quad ||af|| = |a| \cdot ||f|| \quad \text{for any } f, g \in R \text{ and for any complex number } a.$$

(Obviously, from (iii) follows that (iv) $||1|| = 1$ and (v) $||0|| = 0$.)

$||f||$ is called a norm if it also satisfies the condition:

(vi) $||f|| > 0$ if $f \neq 0$. R is called a normed ring if $||f||$ is a norm and if R is complete with respect to the metric $d(f, g) = ||f - g||$.

All norms and quasi-norms discussed in this paper satisfy the following additional condition:

$$(2) \quad ||f^n|| = ||f||^n, \quad n = 1, 2, \dots$$

If $R = \{f\}$ is a ring with a quasi-norm $||f||$ defined on it, then there is a standard way of obtaining a normed ring R^* from it: Let I be the set of all $f \in R$ with $||f|| = 0$. Then I is an ideal in R . Consider the factor ring R/I whose elements are the classes $f^* = f + I$. If we put $||f^*||^* = ||f||$, then $||f^*||^*$ is a norm on R/I , and R^* is obtained from R/I by completing it with respect to the metric $d(f^*, g^*) = ||f^* - g^*||^*$. R^* is called the normed ring obtained from R by identification and completion with respect to the quasi-norm $||f||$, and $\pi^*: f \rightarrow f^*$ is called the natural homomorphism of R into R^* .

3. Let $R = \{f\}$ be a normed ring with the norm $||f||$. We denote by Ω the set of all maximal ideals M of R . For any $f \in R$

and for any $M \in \Omega$, there exists a unique complex number $\alpha = (f, M)$ $= \hat{f}(M)$ such that $f - \alpha \in M$. It is clear that $f \rightarrow (f, M)$ is a continuous homomorphism of R onto the field of complex numbers and satisfies $|(f, M)| \leq \|f\|$. We introduce the weakest topology on Ω with respect to which all functions $\hat{f}(M)$ are continuous on Ω . Ω is a compact Hausdorff space with respect to this topology, and is called the structure space of R .

Let $C(\Omega)$ be the normed ring of all complex-valued functions $\phi(M)$ defined on Ω with the norm:

$$(3) \quad \|\phi\| = \sup_{M \in \Omega} |\phi(M)|.$$

Then $f \rightarrow \hat{f}$ is a continuous homomorphism of R into $C(\Omega)$. This homomorphism is an isometric isomorphism if the norm $\|f\|$ on R satisfies the condition (2). In this case R may be considered as a closed sub-ring of $C(\Omega)$.

For any $f \in R$, the spectrum $\sigma(f, R)$ of f in R is defined as the set of all complex numbers λ such that $f - \lambda$ does not have an inverse in R . It is easy to see that

$$(4) \quad \sigma(f, R) = \hat{f}(\Omega),$$

where $\hat{f}(\Omega)$ denotes the set of values taken by \hat{f} on Ω .

4. We now consider the case $R = B(D)$. We denote by $\Omega(D)$ the structure space of $B(D)$. For each $\beta \in D$, the set $M(\beta)$ of all $f \in B(D)$ with $f(\beta) = 0$ is a maximal ideal of $B(D)$. $M(\beta)$ is called a maximal ideal of $B(D)$ of type I. [All other maximal ideals of $B(D)$ are called maximal ideals of $B(D)$ of type II.] It is easy to see that $\beta \rightarrow M(\beta)$ is a homeomorphism of D into $\Omega(D)$. Thus we may consider D as a subset of $\Omega(D)$ by identifying β with $M(\beta)$. Further, it is easy to see that

$$(5) \quad \sigma(f, B(D)) = Cl[f(D)],$$

where $Cl[E]$ denotes the closure of a set E in the Gaussian plane and $f(D)$ denotes the set of values taken by f on D .

5. Let D_1 and D_2 be two non-empty bounded open sets in the Gaussian plane, and assume that $D_1 \supset D_2$. For any $f \in B(D_1)$, let $\pi(f_1)$ be its restriction to D_2 . Then π is a continuous homomorphism of $B(D_1)$ into $B(D_2)$ and satisfies

$$(6) \quad ||\pi(f_1)||_{D_2} \leq ||f||_{D_1}.$$

π is called the natural homomorphism of $B(D_1)$ into $B(D_2)$. [π is actually an isomorphism if every component of D_1 contains a point of D_2 .] Further,

$$(7) \quad (f_1, \rho(M_2)) = (\pi(f_1), M_2)$$

defines a continuous mapping ρ of $\Omega(D_2)$ into $\Omega(D_1)$. ρ is called the natural mapping of $\Omega(D_2)$ into $\Omega(D_1)$. It is easy to see that if we consider D_1 and D_2 as subsets of $\Omega(D_1)$ and $\Omega(D_2)$, respectively, then the restriction of ρ to D_2 is the identity mapping which embeds D_2 into D_1 . In other words, if $M_2(\beta)$ is a maximal ideal of type I of $B(D_2)$, then $\rho(M_2(\beta))$ is a maximal ideal of type I of $B(D_1)$ which corresponds to the same point $\beta \in D_2$.

6. Let

$$(8) \quad D = D_1 \supset D_2 \supset \dots \supset D_n \supset D_{n+1} \supset \dots$$

be a decreasing sequence of non-empty bounded open sets in the Gaussian plane. Let

$$(9) \quad B(D) = B(D_1) \xrightarrow{\pi_1} B(D_2) \rightarrow \dots \rightarrow B(D_n) \xrightarrow{\pi_n} B(D_{n+1}) \rightarrow \dots$$

be the corresponding sequence of normed rings $B(D_n)$, where π_n is the natural homomorphism of $B(D_n)$ into $B(D_{n+1})$, $n = 1, 2, \dots$.

We define the limit normed ring $R^* = \lim_{n \rightarrow \infty} B(D_n)$ as follows: Let

R be the set of all sequences $f = \{f_n \mid n = 1, 2, \dots\}$ such that

$f_n \in B(D_n)$ for all n and $\pi_n(f_n) = f_{n+1}$ for all sufficiently large n .

(This means that there exists a positive integer $n(f)$ such that

$\pi_n(f_n) = f_{n+1}$ for all $n \geq n(f)$). R becomes a ring if we define:

$af = \{af_n \mid n = 1, 2, \dots\}$, $f+g = \{f_n+g_n \mid n = 1, 2, \dots\}$ and $fg = \{f_n g_n \mid n = 1, 2, \dots\}$ if $f = \{f_n \mid n = 1, 2, \dots\}$ and $g = \{g_n \mid n = 1, 2, \dots\}$.

If we put $\|f\|^* = \lim_{n \rightarrow \infty} \|f\|_{D_n}$, then $\|f\|^*$ is a quasi-norm on R .

The limit normed ring R^* is then obtained from R by identification

and completion with respect to the quasi-norm $\|f\|^*$. We denote

by π^* the natural homomorphism of R into R^* . Then this defines

a homomorphism $\pi_n^*: f_n \rightarrow \pi_n^*(f_n) = \pi^*(f)$ of $B(D_n)$ into R^* ,

where $f = \{f_n \mid n = 1, 2, \dots\}$ is an element of R which satisfies

$f_k = 0$ for $k = 1, 2, \dots, n-1$, and $\pi_k(f_k) = f_{k+1}$ for all $k \geq n$. π_n^*

is called the natural homomorphism of $B(D_n)$ into R^* .

Further, let

$$(10) \quad \Omega(D) = \Omega(D_1) \xleftarrow{\rho_1} \Omega(D_2) \xleftarrow{\quad} \dots \xleftarrow{\quad} \Omega(D_n) \xleftarrow{\rho_n} \Omega(D_{n+1}) \xleftarrow{\quad} \dots$$

be the sequence of structure spaces $\Omega(D_n)$ which corresponds to (9),

where ρ_n is the natural mapping of $\Omega(D_{n+1})$ into $\Omega(D_n)$, $n = 1, 2, \dots$. Since each $\Omega(D_n)$ is compact and since each ρ_n is continuous, we obtain the inverse limit space $\Omega^* = \lim_{n \rightarrow \infty} \Omega(D_n)$ as follows: Ω^* consists of all sequences $M^* = \{M_n \mid n = 1, 2, \dots\}$ such that $M_n \in \Omega(D_n)$ and $\rho_n(M_{n+1}) = M_n$ for all n . Let ρ_n^* be the mapping of Ω^* into $\Omega(D_n)$ defined by $\rho_n(M^*) = M_n$ if $M^* = \{M_n \mid n = 1, 2, \dots\}$. The topology of Ω^* is then defined as the weakest topology on Ω^* with respect to which all mappings ρ_n^* are continuous. It is easy to see that Ω^* becomes a compact Hausdorff space with respect to this topology. ρ_n^* is called the natural mapping of Ω^* into $\Omega(D_n)$, $n = 1, 2, \dots$.

THEOREM 1: Ω^* is the structure space of R^* .

The correspondence between R^* and Ω^* is given as follows: Let M be a maximal ideal of R^* . Then $f^* \rightarrow (f^*, M)$ is a continuous homomorphism of R^* onto the field of complex numbers which satisfies $|(f^*, M)| \leq \|f^*\|$. For each n , $f_n \rightarrow (\pi_n^*(f_n), M)$ is a continuous homomorphism of $B(D_n)$ onto the field of complex numbers and satisfies $|(\pi_n^*(f_n), M)| \leq \|f_n\|_{D_n}$. Consequently, there exists a maximal ideal M_n of $B(D_n)$ such that $(\pi_n^*(f_n), M) = (f_n, M_n)$. It is easy to see that $\rho_n(M_{n+1}) = M_n$ for each n and hence

$M^* = \{M_n \mid n = 1, 2, \dots\}$ determines an element M^* of Ω^* .

Conversely, let $M^* = \{M_n \mid n = 1, 2, \dots\}$ be an element of Ω^* . Then, for each n , $f_n \rightarrow (f_n, M_n)$ defines a continuous homomorphism of $B(D_n)$ onto the field of complex numbers which satisfies $|(f_n, M_n)| \leq \|f_n\|_{D_n}$. It is easy to see that $(f_n, M_n) = (\pi_n(f_n), M_{n+1})$ for any $f_n \in B(D_n)$. Thus for each $f = \{f_n \mid n = 1, 2, \dots\} \in R$, $\lim_{n \rightarrow \infty} (f_n, M_n)$ exists since (f_n, M_n) is independent of n for $n \geq n(f)$. Let us put $(f, M^*) = \lim_{n \rightarrow \infty} (f_n, M_n)$. Clearly, $f \rightarrow (f, M^*)$ is a continuous homomorphism of R onto the field of complex numbers and satisfies $|(f, M^*)| \leq \|f\|$. It is then easy to see that there exists a unique maximal ideal M of R^* such that $(\pi^*(f), M) = (f, M^*)$ for any $f \in R$, where π^* is the natural homomorphism of R into R^* .

THEOREM 2:

$$(11) \quad \sigma(\pi^*(f), R^*) = \bigcap_{n=n(f)}^{\infty} \sigma(f, B(D)) = \bigcap_{n=n(f)}^{\infty} Cl[f_n(D_n)]$$

if $f = \{f_n \mid n = 1, 2, \dots\} \in R$ and $f_n \in B(D_n)$ for all n ,

and $\pi_n(f_n) = f_{n+1}$ for all $n \geq n(f)$.

7. Let D be a non-empty bounded open set in the Gaussian plane, and let ζ be a point from the boundary $\text{Bd}(D)$ of D . Let $f \in B(D)$. A complex number γ^* is called a limit value of f at ζ if there exists a sequence $\{z_n \mid n = 1, 2, \dots\}$ of points from D such that $\lim_{n \rightarrow \infty} z_n = \zeta$ and $\lim_{n \rightarrow \infty} f(z_n) = \gamma^*$. The set of all limit values of f at ζ is called the limit set of f at ζ and is denoted by $\Gamma^*(f, \zeta)$. $\Gamma^*(f, \zeta)$ is a non-empty compact set and is given by

$$(12) \quad \Gamma^*(f, \zeta) = \bigcap_{n=1}^{\infty} \text{Cl}[f(D_n)],$$

where

$$(13) \quad D_n = \{z \mid z \in D, |z - \zeta| < 1/n\}, \quad n = 1, 2, \dots$$

A complex number γ^{**} is called a double limit value of f at ζ if there exist a sequence $\{\zeta_n \mid n = 1, 2, \dots\}$ of points from $\text{Bd}(D)$ and a sequence $\{\gamma_n^* \mid n = 1, 2, \dots\}$ of complex numbers such that $\zeta_n \neq \zeta$, $\gamma_n^* \in \Gamma^*(f, \zeta_n)$ for all n , $\lim_{n \rightarrow \infty} \zeta_n = \zeta$ and $\lim_{n \rightarrow \infty} \gamma_n^* = \gamma^{**}$. The set of all double limit values of f at ζ is called the double limit set of f at ζ and is denoted by $\Gamma^{**}(f, \zeta)$. $\Gamma^{**}(f, \zeta)$ is a non-empty compact set and is given by

$$(14) \quad \Gamma^{**}(f, \zeta) = \bigcap_{\ell=1}^{\infty} \text{Cl} \left[\bigcup_{m=\ell+1}^{\infty} \bigcap_{n=m+1}^{\infty} \text{Cl}[f(\Delta_{\ell, m, n})] \right],$$

where

$$(15) \quad \Delta_{\ell, m, n} = \{z \mid z \in D, d(z, \text{Bd}(D)) < 1/n, 1/m < |z - \zeta| < 1/\ell\}$$

$$n = m+1, m+2, \dots; m = \ell+1, \ell+2, \dots; \ell = 1, 2, \dots$$

The following theorem has been proved by F. Iversen [2],
A. Beurling [1] and K. Kunugui [4]:

THEOREM 3:

$$(16) \quad \Gamma^*(f, \zeta) \supset \Gamma^{**}(f, \zeta),$$

$$(17) \quad \text{Bd}(\Gamma^*(f, \zeta)) \subset \text{Bd}(\Gamma^{**}(f, \zeta))$$

for any $f \in B(D)$.

We shall show that this theorem can be interpreted as a result concerning the spectrum of an element in a normed ring.

8. We now apply the results of section 6 to the case when D_n is given by (13). Theorem 2 shows that the limit set $\Gamma^*(f, \zeta)$ is obtained as the spectrum of a certain element in R^* . The following formulation of the same result is more useful for our purpose:

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Let R' be the ring of all bounded complex-valued functions $f(z)$ defined on D with the property that there exists a positive integer n such that $f(z)$ is analytic on D_n , where D_n is given by (13). R' obviously contains $B(D)$ as a subring. If we put

$$(18) \quad ||f||^* = \overline{\lim}_{z \rightarrow \infty} |f(z)| = \lim_{n \rightarrow \infty} ||f||_{D_n},$$

then $||f||^*$ is a quasi-norm on R' . Let R^* be the normed ring obtained from R' by identification and completion with respect to this quasi-norm $||f||^*$, and let π^* be the natural homomorphism of R' into R^* .

THEOREM 4:

$$(19) \quad \sigma(\pi^*(f), R^*) = \overline{\Gamma}^*(f, \zeta)$$

for any $f \in B(D)$.

9. Let R'' be the ring of all bounded complex-valued functions $f(z)$ defined on D with the following property: there exists a positive integer ℓ such that for any integer $m > \ell$ there exists an integer $n > m$ such that $f(z)$ is analytic on $\Delta_{\ell, m, n}$, where $\Delta_{\ell, m, n}$ is given by (15). R'' obviously contains $B(D)$ as

a subring. If we put

$$(20) \quad ||f||^{**} = \overline{\lim}_{\zeta' \rightarrow \zeta} \overline{\lim}_{z \rightarrow \zeta'} |f(z)| = \lim_{\ell \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} ||f||_{\ell, m, n} \Delta_{\ell, m, n}$$

then $||f||^{**}$ is a quasi-norm on R'' . Let R^{**} be the normed ring obtained from R'' by identification and completion with respect to this quasi-norm $||f||^{**}$, and let π^{**} be the natural homomorphism of R'' into R^{**} .

THEOREM 5:

$$(21) \quad \sigma(\pi^{**}(f), R^{**}) = \Gamma^{**}(f, \zeta)$$

for any $f \in B(D)$.

10. It is now easy to show that Theorem 3 follows from Theorems 4 and 5. We first observe that R' is a subring of R'' , and that

$$(22) \quad ||f||^* = ||f||^{**}$$

for any $f \in R'$. This fact was proved, for example, in K. Kunugui [4]. This means that $\pi^*(f) \rightarrow \pi^{**}(f)$ is an isometric and isomorphic mapping of $\pi^*(R')$ into R^{**} . Since $\pi^*(R')$ is dense in R^* , this

mapping can be extended uniquely to an isometric isomorphism of R^* into R^{**} . Thus we may consider R^* as a closed subring of R^{**} , and Theorem 3 is equivalent to the following well known result in the theory of normed rings:

THEOREM 6: If R^* is a closed subring of a normed ring R^{**} (having the same unit as R^*), then

$$(23) \quad \sigma(f, R^*) \supset \sigma(f, R^{**})$$

$$(24) \quad \text{Bd}(\sigma(f, R^*)) \subset \text{Bd}(\sigma(f, R^{**}))$$

for any $f \in R^*$.

(23) is an immediate consequence of definition, and (24) follows from the fact that, if we denote by G the set of all elements of R^* which have an inverse in R^* , then G is open and every boundary element of G is a generalized null divisor of R^* (C. E. Rickart [5]).

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DERIVATIONS OF BANACH ALGEBRAS

Irving Kaplansky

Derivations of algebras, and in particular Banach algebras, arise in a variety of ways and merit systematic study. In this paper I survey briefly the known results, add some new ones, and mention some open problems.

The first question that appears to have been raised is the following: is every derivation of a commutative semi-simple Banach algebra identically zero? For continuous derivations this was proved by Singer and Wermer. A neat account by Kleinicke [1] is now available. It is based on a purely algebraic lemma.

LEMMA. Let δ be a derivation of a ring A , and x an element in A with $x'' = 0$. Then $(x^n)^{(n)} = n! (\delta x)^n$.

The proof is easily given by induction and Leibnitz's rule.

THEOREM 1. If δ is a continuous derivation of a Banach algebra A , and x is an element with $x'' = 0$, then δx is generalized nilpotent.

PROOF. Let K denote the bound of δ . Then by the lemma,

$$|| (x')^n || \leq K^n || x^n || / n! ,$$

$$|| (x')^n ||^{1/n} \leq K || x^n ||^{1/n} / (n!)^{1/n} .$$

Letting $n \rightarrow \infty$ we see that the right side approaches 0, whence the left side does so too.

The next result was conjectured separately; it is a generalization of a theorem of Jacobson.

THEOREM 2. If in a Banach algebra $ab - ba$ commutes with a , then $ab - ba$ is generalized nilpotent.

PROOF. Let $'$ denote the inner derivation by a . Then $b'' = 0$ and we apply Theorem 1.

THEOREM 3. Let $'$ be a continuous derivation of a Banach algebra A , and x an element of A commuting with x' . Then x' is generalized nilpotent.

PROOF. We pass to the Banach algebra of bounded operators on A . Write D for $'$, R_x for right-multiplication by x . Then $DR_x - R_x D = R_{x'}$ which commutes with R_x . By Theorem 2, $R_{x'}$ is generalized nilpotent, and the same follows for x' .

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COROLLARY. A continuous derivation of a commutative Banach algebra A maps A into the radical.

It is not known whether "continuous" can be omitted in this corollary. We shall pose the question in a still more general form.

PROBLEM I. Is every derivation of a semi-simple Banach algebra continuous?

Only scattered information is available. The answer is affirmative for a commutative C^* -algebra (Singer) and for the algebra of C^n -functions on a manifold (Arnold Shapiro). For the algebra of all bounded operators on a Banach space all derivations are inner and a fortiori continuous.

In all the preceding discussion we have tacitly assumed that the derivation in question is complex linear. It has been useful to study ring automorphisms of Banach algebras, and a decisive result is known. We inquire whether the analogue holds for derivations.

PROBLEM II. Let D be a ring derivation of a semi-simple Banach algebra A . Is it true that A is a direct sum $B \oplus C$ where B is finite-dimensional and D is

linear on C ?

In the final theorem I shall settle both problems for the two simplest algebras of analytic functions. The methods can doubtless be extended to cover more territory (several complex variables, any manifold).

THEOREM 4. Let A be the algebra of all bounded analytic functions in the open disc $|z| < 1$, or the algebra of all functions analytic in the open disc and continuous on its closure. Then any ring derivation of A is 0.

PROOF. Let B denote the algebra of all functions analytic in the open disc (B is not a Banach algebra). The given derivation of A may, if we wish, be regarded as a derivation of A into B . Let D be any derivation of A into B . For any g in B , the mapping $f \rightarrow f'g$ is a derivation of A into B which sends z into g . We may therefore normalize so that $D(z) = 0$. For $f = \sum a_n z^n$ in A we have

$$\begin{aligned} D(f) &= D(a_0) + z (\dots) \\ &= D(a_0) + D(a_1)z + z^2 (\dots), \end{aligned}$$

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etc. Hence $D(f) = \sum D(a_n)z^n$. Now if D is not linear it is discontinuous on the complex numbers. We can find a sequence a_n rapidly approaching 0 such that $D(a_n)$ rapidly approaches ∞ . Then f is in A , but $D(f)$ is not in B , a contradiction. Hence D is linear, and is in fact 0. Returning to our original derivation of A into itself, we find that it has the form $f \rightarrow f'g$ for a suitable g in B . We must now settle the following: supposing that g is not 0, find an f in A such that $f'g$ is not in A . In either of the two cases this is an easy exercise, and we suppress the details.

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THE LAPLACE TRANSFORM ON GROUPS AND GENERALIZED ANALYTIC FUNCTIONS

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1. BACKGROUND

The present note may be regarded as a sequel to another [2] published by the author almost a decade ago. We shall begin by recalling briefly the guiding idea of [2]. Let G be a locally compact abelian group and let \hat{G} be the corresponding dual or character group; that is, the group of all continuous homomorphisms of G into the complex numbers of modulus one. Let \overline{G} be the vector space of all continuous homomorphisms of G into the additive group of the real line. For each $x \in \overline{G}$ the transpose x^* of x is the continuous homomorphism of the real line into \hat{G} defined by the identity $\exp(itx(u)) = x^*(t)(u)$ and the mapping $x \rightarrow x^*$ is one-to-one from \overline{G} onto the group of all such homomorphisms. The following conditions on G are easily seen to be equivalent: (a) \hat{G} is connected, (b) G is the direct product of a vector group and a discrete torsion-free group, (c) the union of the ranges of the homomorphisms $t \rightarrow x^*(t)$ is dense in \hat{G} , (d) the intersection of the kernels of the homomorphisms in \overline{G} is the identity. We shall assume henceforth that one and hence all of these conditions are satisfied.

The most general continuous homomorphism of G into the multiplicative group of all non-zero complex numbers is uniquely of the form $u \rightarrow \exp(x(u))y(u)$ where $x \in \overline{G}$ and $y \in \hat{G}$. The generalized Laplace transform of a (suitably restricted) complex-valued function f on G is the function f on $\overline{G} \times \hat{G}$ (or a subset thereof) defined by the formula: $\hat{f}(x, y) = \int f(u) \exp(x(u))y(u) du$ where du refers to Haar measure in G .

If the vector space G is finite dimensional, as we shall assume from now on, then it has the structure of a C_∞ manifold. In addition \hat{G} has certain essential properties of a C_∞ manifold. Let f be a complex-valued function defined on G and let y_0 and x be points of \hat{G} and \overline{G} respectively. Then f composed with the product of y_0 and x^* is a function of a real variable and we may ask about the existence of its derivative at 0. If this derivative exists we denote it by $f_x(y_0)$. Starting from this notion it is easy to see how to define derivatives of higher order $f_{x_1 x_2 \dots x_n}$ and the notion of C_∞ function. We have a "tangent space" at each point of \hat{G} and this tangent space is naturally isomorphic to \overline{G} . \overline{G} is of course also a universal tangent space for itself regarded as a C_∞ manifold. Similarly one can define C_∞ functions and tangent spaces for the direct product $\overline{G} \times \hat{G}$. Here again there is a universal

tangent space. It is naturally isomorphic to $\overline{G} \oplus \overline{G}$. If we define $i(x_1, x_2)$ as $(-x_2, x_1)$ $\overline{G} \oplus \overline{G}$ becomes a complex vector space. We may thus define analytic functions on $\overline{G} \times \hat{G}$ to be those which are differentiable and have differentials which are complex linear. It turns out that Laplace transforms f are analytic on $\overline{G} \times \hat{G}$ in this sense and that for suitable classes of functions analyticity is equivalent to being a Laplace transform. When G is an n -dimensional vector group, $\overline{G} \times \hat{G}$ is the usual space of n complex variables and the Laplace transform is the classical 2^n -sided Laplace transform in n dimensions. When G is a free abelian group with n generators $\overline{G} \times \hat{G}$ is the direct product of n complex planes with origins missing and the Laplace transform is the classical Laurent series in n variables.

In [2] brief indications were given as to how various classical theorems about analytic functions could be carried over into the more general context just described. Due partly to unexpected difficulties in finding suitably definitive formulations and partly to the pressure of other interests the carrying out of this program was indefinitely postponed and the detailed paper promised in [2] has yet to be written. On the other hand Arens and Singer have recently come upon closely related ideas in studying certain Banach algebras

and have written a paper [1] having some overlap (though from a different viewpoint) with the unwritten development of [2]. They study the completion in a certain norm of the L^1 algebra of a semigroup and show that as an algebra of functions on its maximal ideal space it behaves much like an algebra of analytic functions. From the point of view of [2] this is because this maximal ideal space is the closure of a subset of $\overline{G} \times \hat{G}$ where G is a group generated by their semigroup.

In the present sequel to [2] we propose to outline an expansion of its program in which $\overline{G} \times \hat{G}$ is replaced by a more general object which has no built-in group structure or universal tangent space and of which the complex analytic manifold is a special case.

2. C_∞ AND COMPLEX ANALYTIC ERGODIC LACINGS

Let E be a locally compact Hausdorff space. Let there be given an equivalence relation in E and in each equivalence class the structure of a C_∞ manifold. We shall say that E with this additional structure is a C_∞ ergodic lacing provided that conditions (a), (b) and (c) to follow hold. (a) The identity mapping from each equivalence class (as a C_∞ manifold) to E is continuous.

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(b) Any function from E to the complex numbers which is continuous on E and constant on each equivalence class is constant on E .

(c) For each $p \in E$ there exists a neighborhood N of p (relative to the manifold structure of the equivalence class containing p) such that every C_∞ tensor field defined in N has an extension to a C_∞ tensor field defined on E . In formulating (c) we are anticipating the remark that one defines tangent space at a point and other notions of differential geometry for E in the obvious way making use of the fact that each point of E is contained in a unique C_∞ manifold. We require however that C_∞ functions be continuous in the topology of E as well as in C_∞ on each equivalence class. This requirement of course restricts the class of C_∞ tensor fields on E . An example of a C_∞ ergodic lacing is furnished by the connected group \hat{G} considered above. $x \rightarrow x^*(1)$ is a homomorphism of \overline{G} onto a dense subgroup K of G and K inherits a C_∞ manifold structure by virtue of its isomorphism with a quotient group of \overline{G} . The cosets of K in \hat{G} share this manifold structure. \hat{G} with these cosets as equivalence classes is easily seen to satisfy (a), (b) and (c). Whenever $K \neq G$ we have an example of a C_∞ ergodic lacing which does not reduce to an

ordinary C_∞ manifold. In general, of course, the ordinary C_∞ manifolds are just the C_∞ ergodic lacings in which there is a unique equivalence class. More general examples may be obtained by letting connected Lie groups act ergodically on suitable spaces and taking the orbits as the equivalence classes. We define a complex analytic ergodic lacing as a C_∞ ergodic lacing together with an assignment of "multiplication by i " in the tangent space at each point, this assignment having the following two properties: (a) The tensor field which defines it is a C_∞ tensor field on the underlying C_∞ ergodic lacing. (b) This tensor field converts each equivalence class into a complex analytic manifold. It is clear how one gives the group $\overline{G} \times \hat{G}$ considered above the structure of a complex analytic lacing and how one defines analytic functions on general complex analytic ergodic lacings.

3. INTEGRATION IN C_∞ ERGODIC LACINGS

Let us call an n -th order anti-symmetric covariant tensor in an n -dimensional vector space a determinant. In an oriented C_∞ manifold each C_∞ determinant field defines a unique signed Borel measure in the manifold. The obvious generalization of this correspondence to C_∞ ergodic lacings yields measures which are

infinite on far too many sets and not suitable for our purposes. We shall introduce a different correspondence which, while not so perfect as the classical one, yields better behaved measures and in the case of \hat{G} assigns Haar measure to translation-invariant determinant fields.

We base our discussion on the notion of the divergence of a contravariant vector field with respect to a measure. Let E denote a C_∞ ergodic space and let α be a non-negative Borel measure on E which is finite on compact sets. Let L be a C_∞ contravariant vector field on E . Let $D(E)$ denote the ring of all C_∞ complex-valued functions on E which have compact supports. $L(f)$ then makes sense for each $f \in D(E)$ and $f \rightarrow L(f)$ is a derivation of $D(E)$. We shall say that L has an α -adjoint if for each $g \in D(E)$ there exists a complex valued function g^* in $L^2(E, \alpha)$ such that $\int L(f)g d\alpha = \int fg^* d\alpha$ for all f in $D(E)$. We set $g^* = L^*(g)$ and note that L^* is a linear operator. A simple calculation shows that $f \rightarrow L(f) + L^*(f)$ commutes with multiplication by members of $D(E)$ and hence that there exists a (real-valued) function θ on E such that $L(f) + L^*(f) = -\theta f$ for all f in $D(E)$. We shall call θ the α -divergence of L and write

$\theta = \operatorname{div}_a(L)$. If $\operatorname{div}_a(L)$ is a C_∞ function on E for all C_∞ vector fields L , we shall say that a is a C_∞ measure. We note the computation rules: (a) $\operatorname{div}_a(fL) = f \operatorname{div}_a(L) + L(f)$, (b) $\operatorname{div}_{\rho a}(L) = \operatorname{div}_a(L) + L(\log \rho)$, (c) $\operatorname{div}_a[L, M] = L(\operatorname{div}_a(M)) - M(\operatorname{div}_a(L))$.

Let a_1 and a_2 be C_∞ measures in the same measure class (that is, having the same null sets) so that $a_2 = \rho a_1$. Suppose that $\operatorname{div}_{a_1}(L) = \operatorname{div}_{a_2}(L)$ for all C_∞ vector fields L . It follows from computation rule (b) that $L(\log \rho) = 0$ for all L and hence that ρ is a constant on E . In other words, given the measure class of a , a itself is determined up to a multiplicative constant by the mapping $L \rightarrow \operatorname{div}_a(L)$. Now let W be a nowhere zero C_∞ determinant field in E . Then W defines a C_∞ measure in each equivalence class. Computing the divergence of L with respect to this measure in each equivalence class we get a real-valued function on E which we denote by $\operatorname{div}_W(L)$ and call the divergence of L with respect to W . Of course $\operatorname{div}_W(L)$ may be expressed in purely differential terms. A C_∞ measure a in E will be said to be compatible (with the manifold structure in E) if W exists so that $\operatorname{div}_W(L) = \operatorname{div}_a(L)$ for all C_∞ vector fields L . It is clear from computation rule (b) that every C_∞ measure in the same measure class with a compatible a is also compatible. Hence we may speak of compatible measure classes. Within any such

class each W determines a unique α up to a multiplicative constant. At the moment we can say nothing of interest about the uniqueness of compatible measure classes and can say nothing about their existence in general C_∞ ergodic lacings. On the other hand there are many examples for which it is known that at least one such measure class exists -- the measure class of Haar measure in \hat{G} , for instance. We hazard the conjecture that there are many C_∞ ergodic lacings (other than C_∞ manifolds) in which there exists a unique compatible measure class. In any event one can study the compound system consisting of a C_∞ ergodic lacing together with a particular compatible measure class for it. We shall call such a system a measured C_∞ ergodic lacing.

Let (E, A) denote a measured C_∞ ergodic lacing. Let α_0 be a particular member of A and choose W_0 so that $\text{div}_{\alpha_0}(L) = \text{div}_{W_0}(L)$ for all C_∞ vector fields L . Then $\text{div}_{\rho_{\alpha_0}}(L) = \text{div}_{\rho_{W_0}}(L)$. Thus choosing just one arbitrary constant sets up a one-to-one correspondence between nowhere zero C_∞ determinant fields and members of A . Obviously this correspondence may be extended to one between C_∞ determinant fields, zero or not, and signed measures absolutely continuous with respect to A . When such a choice has

been made we say that (E, A) has been normalized. On a normalized measured C_∞ ergodic lacing every C_∞ determinant function defines a unique signed measure in E .

4. PATH INTEGRALS AND THE CAUCHY THEOREM

Let R be a complex analytic ergodic lacing all of whose tangent spaces have complex dimension n and hence real dimension $2n$. Let E be a compact normalized measured ergodic lacing all of whose tangent spaces have dimension n . By an E path in R we shall mean a C_∞ map from E into R . Let ϕ be an E path in R and let Z be a complex analytic determinant function defined in an open set including the range of ϕ . Then Z composed with the differential of ϕ is of the form $W_1 + iW_2$ where W_1 and W_2 are C_∞ determinant fields in E . We define $\int_\phi Z$ as $\alpha_{W_1}(E) + i\alpha_{W_2}(E)$ where α_W denotes the signed measure in E defined by W . When E is the unit circle, R the complex plane and Z the "differential" $f(z)dz$ then $\int_\phi Z$ reduces to the familiar $\int f(z)dz$ around the closed path defined by ϕ .

Now to a considerable extent the classical Cauchy theorem (and in a weaker sense the Cauchy formula as well) is contained in

the statement that $\int f(z)dz$ does not change when the path of integration is continuously deformed within the domain of analyticity of f . This suggests that one attempt to prove that $\int_{\phi} Z$ is constant under correspondingly restricted continuous deformations of ϕ and thus obtain a Cauchy theorem for general complex analytic ergodic lacings. Our still incomplete work in this direction makes it obvious that such a theorem is true and can be proved without great difficulty. However it is not yet clear exactly what hypotheses are needed and we feel that it would be premature to attempt to formulate a precise result at the present time.

The Cauchy formula described in [2] is a consequence of the Fourier transform theory for groups and the special case of the above in which R is $\overline{G} \times \hat{G}$. E is \hat{G} with the measure class of Haar measure, ϕ_r is of the form $y \rightarrow [r\psi_1(y) + (1-r)\psi_2(y), y]$ and ψ_1 and ψ_2 are C_{∞} functions from \hat{G} to \overline{G} .

It is to be noted that one obtains something more or less new from the general Cauchy theorem described above even when R is the complex plane provided that E is not a C_{∞} manifold. For example if E is \hat{G} where G is the additive group of the rationals then $\phi(G)$ can be the closure of the range of any C_{∞}

almost periodic function on the real line whose Dirichlet expansion involves only terms of the form $c \exp(i\lambda x)$ where λ is rational.

5. GENERAL REMARKS

The preceding is a very preliminary account of our program. At this point we are not even sure that we shall not want to change the definition of C_∞ ergodic lacing to some extent. Ultimately, of course, we shall want to include lacings of the form $\overline{G} \times \hat{G}$ where \overline{G} is infinite dimensional, but infinite dimensionality produces so many new difficulties that it seems best to exclude infinite dimensional tangent spaces for the time being.

At the moment there seems to be some hope of proving that the measure class of Haar measure is invariant under all self homeomorphisms which preserve its structure as a C_∞ ergodic lacing. If so, this measure class can be characterized as the unique one having the indicated invariance property and we may single out as especially interesting those C_∞ ergodic lacings for which such a unique invariant compatible measure class exists. These C_∞ ergodic lacings might well form a larger class than those possessing a unique compatible measure class.

GENERALIZED ANALYTIC FUNCTIONS

For C_∞ manifolds it is known that the tangent space at each point may be described purely algebraically as the set of derivations at that point of the ring of all C_∞ functions with compact support. Whether such a result holds for general C_∞ ergodic lacings is a question which remains to be thoroughly investigated. The classical proof does not extend directly due to the global restriction on C_∞ functions in the general case.

When E is a C_∞ ergodic lacing, but not a C_∞ manifold, one can consider functions which are C_∞ on each equivalence class but instead of being continuous on E have only some weaker property, such as being Borel functions. It seems probable that one can extend the Cauchy theorem to apply to such discontinuous analytic functions.

The lack of a suitable notion including both complex analytic manifolds and lacings of the form $\overline{G} \times \hat{G}$ was one of the chief difficulties which led to our indefinite postponement of the program outlined in [2]. Having now such a notion at our disposal we feel that the chances are excellent that a first detailed paper on the subject will be ready within a year.

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RINGS OF MEROMORPHIC FUNCTIONS

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1. INTRODUCTION

In this paper we discuss some of the algebraic properties of rings of meromorphic functions on Riemann surfaces of finite genus. We consider primarily the compact surfaces and those open surfaces which are open subsets of compact surfaces and have the property that each boundary point (relative to the compact surface) can be an essential singularity for a bounded analytic function on the open surface.

On the compact surfaces we are interested in the field of all meromorphic functions while on the open surfaces we are most interested in the field of quotients of the bounded analytic functions. For the sake of convenience we shall refer to both of these fields as the fields of "meromorphic functions of bounded characteristic".

In Section 3 we characterize the valuations on this field for finite Riemann surfaces, i. e. connected open subsets of compact surfaces whose boundary consists of a finite number of Jordan curves. In the last section we characterize the homomorphisms of this field into the field of all meromorphic functions on an arbitrary Riemann surface.

Some of the present results have been anticipated by Heins [1].

particularly proposition 3 and the corollary to proposition 4. The methods of proof used here are somewhat different and are closer to the work of Rudin [3].

2. SOME FUNCTION-THEORETIC LEMMAS

We list here some lemmas concerning Riemann surfaces. The first is classical [4], and the proof of the third is quite similar. The second lemma is a consequence of Abel's theorem.

LEMMA 1: Let W be a compact Riemann surface and f a meromorphic function on it. Then f assumes each value on the Riemann sphere the same number of times, say n . If g is any other meromorphic function on W , then g satisfies an algebraic equation of degree n whose coefficients are rational functions of f . Moreover, for each complex number z we may choose a meromorphic function g which separates the points $f^{-1}(z)$ and such that every meromorphic function on W is a rational function of f and g .

LEMMA 2: Let W be a non-compact finite Riemann surface. Then for some integer n there is an analytic function

f which maps \overline{W} onto the circle $|z| \leq 1$ so that each value of z in $|z| < 1$ is assumed exactly n times.

LEMMA 3: Let W be a non-compact finite Riemann surface and f the function of lemma 2. Then each analytic function g on W satisfies an equation

$$\sum_{\nu=0}^n b_{\nu}(f) g^{\nu} = 0$$

where the b_{ν} are analytic functions in the interior of the unit circle which are bounded if g is bounded. For each z in $|z| < 1$ we can find a bounded analytic function g on W which separates the points $f^{-1}(z)$ and which has the property that each analytic function h on W can be expressed as

$$h = \sum_{\nu=0}^{n-1} c_{\nu}(f) g^{\nu},$$

where the coefficients c_{ν} are analytic functions in the interior of the unit circle and are bounded if h is.

3. THE VALUATIONS ON SOME FIELDS OF MEROMORPHIC FUNCTIONS

Let F be a field which contains the field C of complex numbers. By a valuation on F we shall mean a mapping v of the non-zero elements of F onto the integers such that

- i) $v(fg) = v(f) + v(g)$
- ii) $v(f + g) \geq \min[v(f), v(g)]$
- iii) If $v(f) \geq 0$, there is a $\lambda \in C$ such that
 $v(f - \lambda) \geq 0$.

It should be noted that these properties imply that $v(\lambda) = 0$ for each non-zero λ in C . This implies that the λ in (iii) is unique.

In this section we prove two propositions about valuations. The first is classical, but the proof is included here both for completeness and for comparison with the proof of the second proposition.

PROPOSITION 1: Let F be the field of meromorphic functions on a compact Riemann surface W . Then for each valuation v on F there is a unique point $p_0 \in W$

such that $v(f)$ is the order of f at p_0 .

PROOF: Let f be a function in F with $v(f) = 1$, and let p_0, p_1, \dots, p_k be the zeros of f . For any rational function r , we have $v[r(f)]$ equal to the order of r at zero. Let g be any function in F . Then g satisfies an equation

$$g^n + \sum_{\nu=0}^{n-1} r_{\nu}(f)g^{\nu} = 0,$$

where the r_{ν} are rational functions of f . If $v(g)$ were negative, then for some ν we have $v(r_{\nu}(f))$ negative, whence r_{ν} has a pole at zero. Since $r_{\nu}(0)$ is an elementary symmetric function of the values of g at the points p_i , we see that g must have a pole at one of the points p_i . Consequently, if g is neither zero nor infinite at any of the points p_i , we have $v(g) = 0$.

Let g be a function which separates the points p_i and whose derivative does not vanish at any of the p_i . Set $g_i = g - g(p_i)$. If ν_i is the order of f at p_i , the function $f / [\prod g_i^{\nu_i}]$ is neither zero nor infinite at any of the p_i and so must have zero valuation. Since $v(g_i) \geq 0$, and

$$1 = v(f) = \sum_{i=0}^k \nu_i v(g_i),$$

we have $v(g_i) = 0$ for all i except one, say $i = 0$, where $v(g_0) = 1$, $v_0 = 1$.

Let h be any element in F and let μ_i be the order of h at p_i . Then $h / [\prod g_i^{\mu_i}]$ is neither zero nor infinite at any p_i and hence has valuation zero. Thus $v(h) = \mu_0$ the order of h at p_0 , proving the proposition.

PROPOSITION 2: Let F be the field of quotients of the bounded analytic functions on a non-compact finite Riemann surface W . Then to each valuation v on F there is a unique point $p_0 \in W$ such that $v(f)$ is the order of f at p_0 .

PROOF: If f is an analytic function such that $|f| < M$ on W , then for each λ with $|\lambda| \geq M$, the function $f - \lambda$ has an n -th root in F for each positive integer n . Thus $v(f - \lambda)$ is divisible by every n and so must be zero. By (ii) we have $v(f) \geq 0$, showing that v is non-negative on the bounded functions.

Let f be the function of lemma 2. Then by (iii) there is a complex number z_0 such that $v(f - z_0) > 0$. Since $v(f - \lambda) = 0$ whenever $|\lambda| \geq 1$, we have $|z_0| < 1$. Let p_0, \dots, p_k be the points on W where f has the value z_0 and let μ_i be the order of $f - z_0$ at p_i . For any bounded function g , the function

$$\frac{\prod (g - g(p_i))^{\mu_i}}{f}$$

is bounded and so $v(g - g(p_i))$ must be positive for some i . Thus if g is a bounded function and $v(g) > 0$, then g vanishes at one of the points p_i .

Let g_0 be a bounded function which separates the p_i . By subtracting a suitable constant we may assume $v(g_0) > 0$. Then g_0 must have a zero at precisely one of the p_i , say p_0 . If g is any bounded function such that $g(p_0) \neq 0$, then for a sufficiently small ϵ the function $g_0 + \epsilon g$ vanishes at none of the points p_i . Thus $0 = v(g_0 + \epsilon g) \geq \min[v(g_0), v(g)]$, whence $v(g) = 0$. Consequently, $v(g) > 0$ implies that $g(p_0) = 0$. If on the other hand $g(p_0) = 0$, we may subtract a constant λ so that $v(g - \lambda) > 0$. But then $g - \lambda$ must vanish at p_0 , whence $\lambda = 0$ and $v(g) > 0$. Thus for bounded functions $v(g) > 0$ if and only if $g(p_0) = 0$.

Now let g be an analytic function on \overline{W} with a simple zero at p_0 and no other zeros on \overline{W} . If h is a bounded analytic function of order ν at p_0 , then $h/(g^\nu)$ is a bounded analytic function which does not vanish at p_0 . Hence its valuation is zero, and $v(h) = \nu v(g)$. Since each function in F is the quotient of two

bounded functions we see that its valuation is just its order at p_0 times $v(g)$. Since v is onto, $v(g) = 1$, and the proposition is proved.

4. HOMOMORPHISMS BETWEEN RINGS OF MEROMORPHIC FUNCTIONS

In this section we discuss some of the consequences of homomorphisms between rings of meromorphic functions on two Riemann surfaces. We first prove a proposition which tells us that in general a ring homomorphism must be an algebraic (or conjugate algebraic) homomorphism.

PROPOSITION 3: Let R_1 and R_2 be any two rings of meromorphic functions on Riemann surfaces W_1 and W_2 , and let ϕ be a ring homomorphism of R_1 into R_2 . If R_1 contains all the complex constants, then ϕ takes constant functions into (possibly different) constant functions. If in addition R_1 contains a bounded function f such that $\phi(f)$ is not constant and if R_1 contains all functions of the form $F \circ f$, where F is analytic on the range of f , then either $\phi(\lambda) = \lambda$ for each complex constant λ or else

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$\phi(\lambda) = \bar{\lambda}$ for each complex constant λ .

PROOF: Unless the range of ϕ is 0, we must have $\phi(1) = 1$.

From this it follows that $\phi(\rho) = \rho$ for each real rational ρ and that $\phi(i)$ is either i or $-i$. For each rational ρ , i.e. a number of the form $\alpha + i\beta$, α and β real rational, we have $\phi(\rho) = \rho$ in the first case and $\phi(\rho) = \bar{\rho}$ in the second. If λ is any complex constant, then $\lambda - \rho$ has a square root for any rational ρ , and thus for each rational ρ the function $\phi(\lambda) - \rho$ has a square root. Consequently $\phi(\lambda)$ must be constant.

Let f be the bounded analytic function given in the second part of the proposition. Without loss of generality we may assume that 0 is in the range of $\phi(f)$. Since $\phi(f)$ is not constant there must be a positive δ such that all numbers whose modulus is less than δ are in the range of $\phi(f)$. Let M be the supremum of $|f|$, and for any complex constant λ take a real rational ρ such that $\rho M |\lambda| < 1$. Then the function $g = 1 + \rho \lambda f$ has an n -th root for every n . Consequently the function $\phi(g) = 1 + \rho \phi(\lambda) \phi(f)$ must also have an n -th root for every n . But this implies that $[\rho \phi(\lambda)]^{-1}$ is not in the range of $\phi(f)$, and so $\rho |\phi(\lambda)|$ must be less than $1/\delta$, whence

$$|\phi(\lambda)| < \frac{1}{\delta \rho}.$$

Since this is true for all positive rational ρ which are less than $M(|\lambda|)^{-1}$, we have

$$|\phi(\lambda)| \leq \frac{M|\lambda|}{\delta}.$$

Thus ϕ is a continuous mapping of the complex field onto itself, and so we must have $\phi(\lambda) = \lambda$ for all λ or else $\phi(\lambda) = \bar{\lambda}$ for all λ .

PROPOSITION 4: Let W_1 be a Riemann surface and F_1 a field of meromorphic functions on it with the property that for each valuation v on F_1 there is a unique point $p \in W_1$ such that $v(f)$ is the order of f at p . Then for any algebraic isomorphism ϕ of F_1 into the field F_2 of all meromorphic functions on a Riemann surface W_2 , there is an analytic mapping ψ of W_2 into W_1 such that $\phi f = f \circ \psi$.

PROOF: Since ϕ is one-to-one, the range of ϕ contains non-constant functions. Consequently, for each $p \in W_2$ the function $W(f)$ which gives the order of $\phi(f)$ at p is a mapping of F_1 into the integers which is either a valuation on F_1 or a positive integer times a valuation. By hypothesis there is a point $\psi(p) \in W_1$

such that this valuation is just the order of a function at $\psi(p)$. Thus $\phi(f)$ vanishes at p if and only if f vanishes at $\psi(p)$. Consequently $\phi f = f \circ \psi$. The mapping ψ is readily seen to be analytic, proving the proposition.

COROLLARY: Let W_0 be a compact Riemann surface and F_0 its field of meromorphic functions. Then for any algebraic isomorphism ϕ of F_0 into the field F_2 of all meromorphic functions on a Riemann surface W_2 there is an analytic mapping ψ of W_2 into W_0 such that $\phi(f) = f \circ \psi$.

THEOREM 1: Let W_0 be a compact Riemann surface, and W_1 a connected dense open subset of W_0 with the property that for each boundary point of W_1 there is a bounded analytic function on W_1 having an essential singularity at that point. Let ϕ be a ring isomorphism of the ring R_1 of meromorphic functions of bounded characteristic on W_1 into the ring R_2 of all meromorphic functions on some Riemann surface W_2 . Then if $\phi(i) = i$, there is an analytic mapping ψ of W_2 into W_1 such that

$$\phi(f) = f \circ \psi.$$

PROOF: By proposition 4, $\phi(\lambda) = \lambda$ for each complex constant λ , and we have an algebraic isomorphism.

We first note that the field F_0 of meromorphic functions on W_0 is contained in F_1 . For let p_0 be a fixed point of W_1 . Then each f in F_0 is a rational function of functions in F_0 which poles only at p_0 . But if $f \in F_0$ has a pole of order n at p_0 and no other poles on W_0 and if g is a bounded analytic function on W_1 , then $[g - g(p_0)]^n f$ is also a bounded analytic function on W_1 , whence f is the quotient of two bounded analytic functions on W_1 .

Thus ϕ maps F_0 into F_2 , and by the corollary of proposition 4, there is an analytic mapping ψ of W_2 into W_0 such that $\phi(f) = f \circ \psi$ for $f \in F_0$.

We next note that ϕ carries bounded functions into bounded functions. For if $f \in F_1$ is bounded by M , then $f - \lambda$ has an n -th root whenever $|\lambda| \geq M$. But this implies $\phi(f) - \lambda$ has an n -th root whenever $|\lambda| \geq M$, whence $|\phi(f)| < M$ on W_2 .

Let f be a bounded function in F_1 , and $p \in W_2$ be a point such that $\psi(p) \in W_1$. Let $g \in F_0$ be a function in F_0 which is analytic except at $\psi(p)$ where it has a pole of order n . Then

$\{f - f[\psi(p)]\}^n g$ is a bounded function and so its image under ϕ must also be bounded. But this image is $\{\phi(f) - f[\psi(p)]\}^n \phi(g)$, and $\phi(g)$ has a pole at p . Hence at p we must have $\phi(f) = f[\psi(p)]$.

Thus $\phi(f) = f \circ \psi$ in $\psi^{-1}(W_1)$. Since $\psi^{-1}(W_1)$ is a non-empty open set, this implies that each bounded analytic function in W_1 is also analytic in $\psi(W_2)$, and so we must have $\psi(W_2) \subset W_1$. This completes the proof of the theorem.

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1. It is well known that the theory of locally convex algebras is not a trivial generalization of Banach theory. The inverse a^{-1} is defined and continuous on a neighborhood of the unity of A , if A is a Banach algebra, but need not be defined, much less continuous, if A is only supposed locally convex. This property is important for the proof of many important theorems of Banach theory. We cannot hope to generalize these to general locally convex algebras.

I shall try to show how important this continuous inverse property is in Banach theory, by generalizing parts of Banach algebra theory to locally convex algebras where the inverse has stronger or weaker continuity properties. I shall also try to indicate the relationship which exists between the properties of the inverse, and the parts of the theory that can be generalized in this fashion.

This program is too ambitious for the space allotted to me. I shall study here the theory of analytic functions on locally convex commutative algebras. The relationship existing between the properties of the inverse and the more general properties of the algebra appear clearly in that theory. Also, these relationships are typical,

and other parts of Banach theory can be generalized in a similar way to locally convex algebras with stronger or weaker continuity properties for the inverse.

2. The algebra A which we shall study is commutative with unity over the complex field. A locally convex topology will be defined on A . That topology will be supposed complete, and the product continuous in both variables.

These topological hypotheses are somewhat stronger than we need.⁽¹⁾ We shall not try to weaken them, since we wish to place the emphasis on the continuous inverse hypothesis, and not on the continuous product hypothesis.

The algebra A has a unity element. We shall identify it with the complex number 1, and its product by the complex number z with z itself. This identification is classical, makes for simpler notations, and will not be misleading below.

Bounded sets will have to be considered in A . Those are the sets which can be sent homothetically into all neighborhoods of

(1)

See L. Waelbroeck. Note sur les algèbres du calcul symbolique. Journal de Mathématiques pures et appliquées. (To be published).

zero. Such a set has the following property: za tends uniformly to zero when z is a complex number which tends to zero, and a varies over the set.

3. Let f be a function with domain D in A , and with range in A . The function f is A-differentiable if its differential in some sense is A-linear. We shall wish to be as general as possible at the outset, and shall therefore consider the Gâteaux differential.⁽²⁾

$f(z)$ is A-differentiable in the sense of Gâteaux at a point z_0 in A , if we can find an A-linear function $u(h)$, which is such that

$$\frac{f(z_0 + sh) - f(z_0)}{s} \rightarrow u(h)$$

when s is a complex number which tends to zero.

Suppose D is open; $f(z)$ is analytic over D if it is A-differentiable in the sense of Gâteaux at each point of C .⁽³⁾

(2) Gâteaux. Fonctions d'une infinité de variables indépendantes. Bull. Soc. Math. Paris, 47. 1919; Sur diverses questions de calcul fonctionnel. Bull. Soc. Math. Paris. 50. 1922.

(3) This generalizes the definition given by E. R. Lorch, in "Analytic functions on vector rings", Trans. Am. Math. Soc. 54. 1943, when A is a Banach algebra.

The A-linear functional can then be set equal to $hf'(z)$,

$$\left[\frac{d}{ds} f(z+sh)\right]_{s=0} = hf'(z)$$

for all z in D , and all h in A , if s is a complex variable. The function $f'(z)$ is clearly analytic over D . We can define successive derivatives, i. e. $f^{(r)}(z)$ for all positive r .

The Taylor series

$$f(z+h) = f(z) + hf'(z) + \frac{h^2}{2!} f''(z) + \dots$$

converges if h is an element of A such that $(z+sh) \in D$ for all complex s such that $|s| \leq 1$. The set of those h is a neighborhood of zero; the sum of the series is $f(z+h)$. Applying the classical proof, we show that an analytic function vanishes identically over D , if D is connected, and all derivatives of f vanish at one point of D .

4. Let A be an algebra, and D be any open set in A .

We can always find functions which are analytic on D : the polynomials, for example. There may also be analytic functions which are not polynomials.

One may ask whether there are always analytic functions on D which are not polynomials. The answer is negative. We shall construct an algebra which is such that f is a polynomial if f is analytic on a domain of this algebra.

The elements of this algebra A are polynomials in one variable x with complex coefficients. The topology of A is the finest locally convex topology which can be defined on that space. Addition and multiplication are defined in the usual way. This algebra has the properties listed at the beginning of this paper; one can show that it is complete, and that the product defined on A is continuous in the two variables.

Let then $f(z)$ be analytic when z is near to zero in A . The Maclaurin expansion

$$f(z) = \sum_r a_r z^r$$

converges when z is near zero, converges therefore when $z = sx$, if s is a small complex number:

$$f(sx) = \sum_r a_r s^r x^r.$$

The sum of this series belongs to A , hence is a polynomial (of

finite degree) in x . The power series can thus have only a finite number of non-vanishing terms, and the function $f(z)$ is a polynomial.

5. We cannot say that the theory of analytic functions on the algebra A considered above is a good generalization of the theory of analytic functions on Banach algebras. The following proposition is true when A is a Banach algebra:

There are locally as many analytic functions of an A -variable as there are A -valued functions of a complex variable. Suppose $f(s)$ is a function of the complex variable s , which is analytic for s near zero. We can find a function $F(z)$ of an A -variable (and obviously only one, locally) which is analytic for z near zero, and such that

$$f(s) = F(s)$$

when s is a small complex number. (We identify the complex number s , with the product by s of the unity element of A .)

As we have just said, this is true when A is a Banach algebra. It is not true when A is the algebra defined above. Other-

wise, $(1-z)^{-1}$ would be analytic for z near to zero, and it is not analytic there. (It is not defined there.)

In fact, suppose the analytic extension theorem true; $(1-s)^{-1}$ is an analytic function of the complex variable s , when s is near zero. It has therefore an analytic extension, $F(z)$. This extension is analytic over a neighborhood of zero and satisfies the equation $(1-z)F(z) = 1$ when z is complex, and therefore for all z in the domain of $F(z)$ since there can only be one analytic function of z which is equal to 1 when z is complex.

We show that $F(z)$ is an inverse of $(1-z)$ over its domain, i. e. $(1-z)^{-1}$ is defined and analytic for z near zero, and therefore z^{-1} when z is near the unity.

6. We shall say that there are enough analytic functions on an algebra A if the local extension proposition is true. We have seen that z^{-1} is then defined and analytic for z near the unity.

Conversely, suppose that z^{-1} is defined and analytic when z is near the unity. Let $f(s)$ be an analytic function of the complex variable s , when s is near zero.

$$f(s) = \frac{1}{2\pi i} \int_j \frac{t^{-1}f(t)}{1-t^{-1}s} dt$$

if j is a circle of sufficiently small radius, and center zero, described positively, and if s is a point inside of j .

The function $(1-t^{-1}z)^{-1}$ is an analytic function of z , for z in A , near zero, and t on j .

$$F(z) = \frac{1}{2\pi i} \int_j t^{-1} f(t) [1 - t^{-1}z]^{-1} dt$$

is an analytic extension of z , since the integral converges uniformly. This function is an extension of $f(s)$. There will therefore be enough analytic functions on A , if a^{-1} is analytic for a near the unity.

7. Let us now seek conditions for the analyticity of z^{-1} . We know that $(1+sh)^{-1}$ is an analytic function of the complex number s , if h is in A , and s small (depending on h), if the function z^{-1} is analytic for z near the unity. Analytic functions of a complex variable are continuous, and therefore bounded over some neighborhood of the origin: $(1+sh)^{-1}$ is therefore bounded when $|s| \leq \epsilon(h)$.

We shall say that an $a \in A$ is regular⁽⁴⁾ if $(a-t)^{-1}$ is

⁽⁴⁾ Defined in L. Waelbroeck. Le calcul symbolique dans les algèbres commutatives. Journal de Math. Pures et App., (9), V. 33, (1954). pp. 147-186.

defined and bounded for $|t| \geq M$. All elements of A are regular if z^{-1} is analytic for z near the unity. The inverse is furthermore defined on a neighborhood of the unity, if it is analytic there.

8. Conversely, suppose that the inverse is defined over some neighborhood of unity, and suppose further that all elements of A are regular, i. e. that $(a-t)^{-1}$ is bounded for $|t| \geq M(a)$. Let h be an element of A , and s a small complex number.

$$(1+sh)^{-1} = 1 - h(h+s^{-1})^{-1}$$

is defined and bounded if $|s| < M(h)^{-1}$,

$$(1+sh)^{-1} - 1 = -sh(1+sh)^{-1}$$

tends to zero when the complex number s tends to zero. (The right-hand side is the product of two bounded factors by a complex number which tends to zero.)

We therefore see that

$$\frac{(1+sh)^{-1} - 1}{s} + h = h[-(1+sh)^{-1} + 1]$$

tends to zero when s tends to zero. The function z^{-1} has an

A-derivative in the sense of Gâteaux for $z = 1$, and this derivative is equal to -1 .

With a simple change of variables, applying the identity

$$z^{-1} = a^{-1} [a^{-1} z]^{-1}$$

we finally see that

$$\frac{dz^{-1}}{dz} = -z^{-2}$$

on the set of invertible z 's. This set is open. The inverse has an A-derivative on an open set; it is analytic on that set.

The inverse will be analytic if and only if all elements of A are regular, the set of invertible elements being a neighborhood of the unity. This is therefore a necessary and sufficient condition that there be "enough" analytic functions on A .

9. The following is an interesting special case: we shall say A is a continuous-inverse algebra ⁽⁵⁾ if the set of invertible elements is a neighborhood of the identity element, and the inverse is continuous for $a = 1$.

⁽⁵⁾ L. Waelbroeck. Les algèbres à inverse continu. C.R. Paris, t. 238 (1954).

It is well known that the set of invertible elements is then open, and that a^{-1} is continuous for all invertible a . Further $(1+sh)^{-1}$ is bounded if $|s| < \epsilon(h)$, and all elements of A are therefore regular. The inverse is analytic, therefore we have "enough" analytic functions.

In the Cauchy integral expansion (paragraph 6), the integrand is a continuous function of z . The integral is continuous too. Analytic functions on continuous-inverse algebras are continuous. We see here how the properties of the inverse, and those of the analytic functions on A can be linked, and how intermediate continuity conditions for the inverse yield intermediate conditions for the analytic functions.

Other properties of Banach algebras can be generalized too. The quotient of a continuous-inverse algebra by a closed ideal is a continuous-inverse algebra, of course. It is somewhat more difficult to show that a closed subalgebra of a continuous-inverse algebra is a continuous-inverse algebra. We must show that the elements of the subalgebra A' which are near the unity have their inverse in A' . But the Maclaurin expansion of $(1-h)^{-1}$ is a geometric series. The terms of the series all belong to A' ; the sum must belong to A' too.

The maximal ideals of continuous-inverse algebras are obviously closed. (The set of invertible elements is a neighborhood of the unity.) The quotient A/m is a continuous-inverse locally convex field. Such a field is isomorphic to the complex number field. ⁽⁶⁾.

These two properties of maximal ideals of A are essentially those needed to generalize the Gelfand theory of maximal ideals to continuous-inverse algebras.

We can generalize in this way an important part of Banach algebra theory to continuous-inverse algebras. Weaker continuity conditions for the inverse, besides, allow weaker generalizations of Banach theory. We have shown this in analytic function theory, but it is true in other parts of Banach theory too. The possible generalizations, though, are not as satisfactory in that weaker case, as they are in the continuous inverse case.

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⁽⁶⁾ L. Waelbroeck. Les surcorps du corps des nombres réels. Bull. Acad. de Belgique (5) 38, (1952).

RINGS OF ANALYTIC FUNCTIONS

John Wermer

Let E be the open unit disk in the z -plane and \overline{E} its closure. Let A be the ring of all functions continuous on \overline{E} and analytic on E . We use on A the topology of uniform convergence on \overline{E} . Let R be a subring of A which contains all constants and separates points in \overline{E} , i. e.

(1) If $z_1 \neq z_2$ with $z_1, z_2 \in \overline{E}$, then there is some g in R with $g(z_1) \neq g(z_2)$.

QUESTION: Under what conditions is R dense in A ?

A well-known theorem of Walsh answers this question in the case when R is the ring of polynomials in a single function ϕ . In this case (1) is necessary and sufficient in order that R be dense in A .

In the general case, the condition

(2) If $z \in E$, there is some g in R with $g'(z) \neq 0$,

is obviously necessary for R to be dense in A . Also (2) is not implied by (1) as is seen by taking for R the ring of all functions in A whose derivative vanishes at the origin.

However (1) and (2) together are not sufficient in general. To see this we take a simple closed curve β having on it an arc α of positive plane measure. Let R_1 be the ring of all functions analytic in the interior B of β which can be extended to the whole Riemann sphere to be everywhere continuous and to be analytic on the complement of α . By reasoning given in [1] we can verify that (1) and (2) hold for R_1 relative to $B \cup \beta$. Let now τ be a conformal map of B on E . Then τ extends to a homeomorphism of $B \cup \beta$ onto \bar{E} . This homeomorphism carries R_1 onto a subring R of A which satisfies (1) and (2). If R were dense in A , then R_1 would be dense in the ring A_1 of all functions analytic on B and continuous on $B \cup \beta$. But R_1 is a closed subring of A_1 , by the maximum principle applied to the complement of the arc α on the sphere. Also clearly $R_1 \neq A_1$. Hence R_1 is non-dense in A_1 and so R is non-dense in A .

Let us now assume the following:

(3) Every function in R is analytic on the closed disk \bar{E} .

(4) There is some $\phi_0 \in R$ with $\phi'_0 \neq 0$ on $|z| = 1$.

THEOREM 1: Let R be a subring of A satisfying (3) and (4). Then R is dense in A if and only if conditions (1) and

(2) hold.

NOTE: This theorem remains true when the unit disk is replaced by any finite Riemann surface whose boundary is a simple closed analytic curve, provided we replace "derivative" by "differential" in (2) and (4). The proof is practically the same as that of Theorem 1.

If on the other hand we replace the unit disk by a region with more than one boundary curve, Theorem 1 is no longer true. For instance, let A_2 be the ring of functions analytic in the open annulus $1 < |z| < 2$ and continuous in the closed annulus and let R_2 be the ring of polynomials in z . Then clearly R_2 satisfies (1), (2), (3), (4) relative to the annulus and yet R_2 is non-dense in A_2 .

We shall derive Theorem 1 from the following known result:
([2], Chap. IX, p. 10, Lemma)

LEMMA 1: Let V be a complex-analytic variety, K a compact set in V and P an algebra of functions analytic in V which contains all constants. Assume

(5) If $x \notin K$, there is a g in P with $|g(x)| > \sup_{y \in K} |g(y)|$.

(6) The functions in P separate points in V .

(7) Each point in V has a system of local coordinates

consisting of functions in P .

Then every function analytic on a neighborhood of K can be uniformly approximated on K by functions in P .

Observe that conditions (6) and (7) are very nearly the same as (1) and (2) if we take for V a disk of radius $1 + \epsilon$, $\epsilon > 0$. The crucial point in the proof of Theorem 1 is thus the verification of (5). We begin with:

LEMMA 2: Let R satisfy (1), (2), (3), (4) on \bar{E} . Then we can find a set of four functions f_i in R , $i = 1, \dots, 4$ and a disk V : $|z| < 1 + \epsilon$, $\epsilon > 0$, such that each f_i is analytic on V , and such that:

(8) If $a \in V$, $|a| > 1$ and $b \in V$, $|b| \leq 1$, then $f_1(a) \neq f_1(b)$ or $f_2(a) \neq f_2(b)$.

(9) The f_i separate points in V .

(10) For each $x \in V$ there is some i with $f_i'(x) \neq 0$.

PROOF: We first show that there is a function f_1 in R taking only finitely many values more than once on $|z| = 1$ and with non-vanishing derivative on $|z| = 1$. It is then an immediate consequence of (1) that there is some f_2 in R , f_2 not a constant, such that f_1 and f_2 together separate points on $|z| = 1$. We can

then show that there exist at most finitely many pairs (p, q) of distinct points in $|z| \leq 1$ such that $f_1(p) = f_1(q)$ and $f_2(p) = f_2(q)$. Because of (1) we can then find a function f_3 in R such that f_1, f_2, f_3 together separate points in $|z| \leq 1$. Next f_1' can vanish at only finitely many points in $|z| < 1$. Because of (2) we can then find some f_4 in R with $f_4' \neq 0$ at each of these points.

The functions f_1, f_2, f_3, f_4 then together separate points in $|z| \leq 1$ and at each point of $|z| \leq 1$ at least one of them has a non-vanishing derivative. Also $f_1' \neq 0$ on $|z| = 1$. Because of (3) there is a disk: $|z| < 1 + \delta, \delta > 0$, in which each $f_i, 1 \leq i \leq 4$, is analytic.

From the preceding it easily follows now that some subdisk V_0 ,

$|z| < 1 + \varepsilon_0, \varepsilon_0 > 0$, satisfies (9) and (10). Finally, since f_1 and f_2 together separate points on $|z| = 1$ and $f_1' \neq 0$ on $|z| = 1$, one can prove that there is some $\varepsilon, 0 < \varepsilon < \varepsilon_0$ such that in the region

$1 < |z| < 1 + \varepsilon$ there exists no point a which is identified by both

f_1 and f_2 with some point b in $|z| \leq 1$. We take V to be the disk:

$|z| < 1 + \varepsilon$. Then (8) holds for V , and clearly so do (9) and (10), as

V is contained in V_0 . Thus V is the required disk.

Let f_1, f_2 be the functions introduced in the proof of Lemma 2.

We write $[f_1, f_2]$ for the ring of polynomials in f_1, f_2 and $C[f_1, f_2]$

for the uniform closure of $[f_1, f_2]$ on $|t| = 1$. Each g in $C[f_1, f_2]$

is then the restriction to $|t| = 1$ of a unique function continuous on \bar{E} and analytic on E . We denote that function again by g . Let m be a non-zero multiplicative linear functional on $C[f_1, f_2]$. We say that m is induced by a point x in \bar{E} if for all g in $C[f_1, f_2]$

$$(11) \quad m(g) = g(x), \text{ or, equivalently, } m(f_1) = f_1(x) \\ \text{and } m(f_2) = f_2(x).$$

LEMMA 3: Assume m is induced by no point on $|z| = 1$.

Then we can find ε , with $|\varepsilon|$ arbitrarily small, such that

if $\phi = f_1 + \varepsilon f_2$ then $m(\phi)$ does not lie on $\{\phi(t): |t| = 1\}$.

PROOF: Set $X(t) = \frac{m(f_1) - f_1(t)}{m(f_2) - f_2(t)}$. Then X is meromorphic

on $|t| = 1$. Hence we can find arbitrarily small ε with $X(t) \neq -\varepsilon$ for all t on $|t| = 1$. Set $\phi = f_1 + \varepsilon f_2$ for such an ε . Assume for some t_0 on $|t| = 1$ we have $m(\phi) = \phi(t_0)$. Then

$$m(f_1) + \varepsilon m(f_2) = f_1(t_0) + \varepsilon f_2(t_0)$$

or

$$m(f_1) - f_1(t_0) = -\varepsilon(m(f_2) - f_2(t_0)).$$

If $m(f_2) = f_2(t_0)$, then $m(f_1) = f_1(t_0)$. Hence m is induced by t_0 , contrary to hypothesis. Thus $m(f_2) \neq f_2(t_0)$, whence $X(t_0) = -\varepsilon$.

This is a contradiction. Hence $m(\phi) \notin \{\phi(t): |t| = 1\}$. Q.E.D.

LEMMA 4: Every non-zero multiplicative linear functional M on $C[f_1, f_2]$ is induced by some point of \bar{E} .

PROOF: Assume M is induced by no point of $|z| = 1$. We shall show that M is induced by some point of $|z| < 1$.

By Lemma 3 we can choose ϵ such that $\phi = f_1 + \epsilon f_2$ has the property that $M(\phi) \neq \phi(t)$ for all t on $|t| = 1$. Also by choice of f_1 , $f_1' = 0$ on $|t| \neq 1$ and f_1 takes only finitely many values more than once on $|t| = 1$. For $|\epsilon|$ sufficiently small ϕ retains these properties, and we may hence assume

$$(12) \quad \phi' \neq 0 \text{ on } |t| = 1.$$

$$(13) \quad \phi \text{ takes only finitely many values more than once on } |t| = 1.$$

Also f_1 and f_2 together separate points on $|t| = 1$. Hence

$$(14) \quad \phi \text{ and } f_2 \text{ together separate points on } |t| = 1.$$

If we define $[\phi, f_2]$ and $C[\phi, f_2]$ in analogy with our earlier definitions for f_1 and f_2 we get

$$(15) \quad [\phi, f_2] = [f_1, f_2], \quad C[\phi, f_2] = C[f_1, f_2].$$

Let now γ be the image of $|t| = 1$ under ϕ and let Ω be the complement of γ in the plane. Let $d\mu$ be a complex measure on $|t| = 1$ with

$$(16) \quad \int_{|t|=1} g(t) d\mu(t) = 0, \quad \text{all } g \text{ in } [\phi, f_2]$$

(17) $d\mu$ has no point mass at any point on $|t| = 1$ which ϕ sends into a multiple point on γ .

Let W be a component of Ω . Denote by $\phi^{-1}(W)$ the region on $|z| < 1$, (which may be empty) which ϕ maps on W . Then each point on W is covered m times, where m is some integer depending on W , except for a possible finite set of branch-points.

One can now prove the following: there exists a unique meromorphic function k on $\phi^{-1}(W)$ which depends on $d\mu$, such that for each z in W , z not a branch-point, we have for all g in $C[\phi, f_2] = C[f_1, f_2]$:

$$(18) \quad \sum_{i=1}^m g(p_i) k(p_i) = \frac{1}{2\pi i} \int_{|t|=1} \frac{g(t) d\mu(t)}{\phi(t) - z}$$

where p_1, \dots, p_m are the points in $\phi^{-1}(W)$ which ϕ maps on z .

If $m = 0$, we take the left side in (18) to be zero. Further, there exists an integer $n(W)$ independent of the measure $d\mu$ such that the poles of k in $\phi^{-1}(W)$ are of order $\leq n(W)$.

Formula (18) and the assertions regarding k are proved by a lengthy argument based on (12), (13), (14). This proof is given in [3], Theorem 2.1. Here I only wish to show how (18) may be used to prove Lemma 4.

Since m is induced by no point on $|t| = 1$ we can find η in $C[f_1, f_2]$ such that $M(\eta) = 1$ and $\eta(t) = 0$ at each t on $|t| = 1$ which ϕ takes on a multiple point of γ .

By choice of ϕ , $M(\phi)$ lies in some component V of Ω . Assume first $\phi^{-1}(V)$ has m sheets, $m \neq 0$. Let $n(V)$ be the integer attached to V above. Let q_1, \dots, q_ℓ be the points in $\phi^{-1}(V)$ which ϕ maps on $M(\phi)$. Assume M is not induced by any one of the q_i . We shall obtain a contradiction from this.

For then we can find some g_0 in $C[f_1, f_2]$ with $M(g_0) = 1$ and $g_0(q_i) = 0$, $i = 1, \dots, \ell$. Hence if $g_1 = g_0^{n(V)+1}$, $g_1 \in C[f_1, f_2]$, $M(g_1) = 1$ and g_1 has at each q_i a zero of order $\geq n(V) + 1$.

Let now $d\sigma$ be any measure orthogonal to $C[f_1, f_2]$ on $|t| = 1$. Then $\eta d\sigma$ is a measure with the same property and so satisfies (16). Also $\eta d\sigma$ satisfies (17), because of the choice of η .

Let K be that meromorphic function on $\phi^{-1}(V)$ attached to the measure $\eta d\sigma$ which obeys (18). Then (18) asserts for $g = g_1$:

$$(19) \quad \sum_{i=1}^m g_1(p_i) K(p_i) = \frac{1}{2\pi i} \int_{|t|=1} \frac{g_1(t) \eta(t) d\sigma(t)}{\phi(t) - z},$$

where z is any point of V not a branch-point and p_1, \dots, p_m lie in $\phi^{-1}(V)$ over z . Whether or not $M(\phi)$ is a branch-point, (19) yields

$$(20) \quad \frac{1}{2\pi i} \int_{|t|=1} \frac{g_1(t) \eta(t) d\sigma(t)}{\phi(t) - M(\phi)} = \lim_{z \rightarrow M(\phi)} \sum_{i=1}^m g_1(p_i) K(p_i).$$

Since K has only poles of order $\leq n(V)$ and g_1 has zeros of order $\geq n(V) + 1$ at the q_i , the right-hand limit is zero. Hence the measure $d\sigma$ is orthogonal to the function

$$(21) \quad \frac{g_1 \eta}{\phi - M(\phi)} = h_1.$$

But $d\sigma$ was an arbitrary measure orthogonal to $C[f_1, f_2]$ on $|t| = 1$.

Hence the function h_1 on $|t| = 1$ defined by (21) lies in $C[f_1, f_2]$.

We have

$$(22) \quad g_1 \eta = h_1 (\phi - M(\phi)).$$

Hence $M(g_1)M(\eta) = M(h_1) \cdot 0 = 0$. But $M(g_1) = 1$ and $M(\eta) = 1$, by

choice of g_1 and η . This is impossible. Hence M must be induced

by one of the q_i , and so by a point in $|z| < 1$.

Thus the Lemma is proved, provided $\phi^{-1}(V)$ is non-empty.

If $\phi^{-1}(V)$ is empty, (18) gives

$$(23) \quad 0 = \int_{|t|=1} \frac{g d\mu}{\phi - M(\phi)}, \quad \text{all } g \text{ in } C[f_1, f_2],$$

if $d\mu$ satisfies (16), (17).

If $d\sigma$ and η are as above, we find that $d\sigma$ is orthogonal to

$$h_2 = \frac{\eta}{\phi - M(\phi)}$$

whence, reasoning as before, we get $M(\eta) = 0$, which is false.

Hence $\phi^{-1}(V)$ cannot be empty. The Lemma thus holds.

PROOF OF THEOREM 1: The necessity of (1) and (2) is clear.

Assume (1), (2), (3) and (4) hold for R . Let V be the disk of Lemma

2 and K the closed unit disk and let P be the algebra of all poly-

nomials in f_1, \dots, f_4 , where these are the functions of Lemma 2.

Because of (9) and (10), P satisfies hypotheses (6) and (7) of Lemma 1.

We assert that also hypothesis (5) is satisfied. For assume the contrary. Then there is some $x \in V$, $|x| > 1$, such that

$$(24) \quad |g(x)| \leq \sup_{|y| \leq 1} |g(y)|, \quad \text{all } g \text{ in } P.$$

In particular this holds for g in $[f_1, f_2]$. From this it follows by an easy limit argument that there exists a linear multiplicative functional M on $C[f_1, f_2]$ with

$$(25) \quad M(g) = g(x), \text{ all } g \text{ in } [f_1, f_2].$$

By Lemma 4 we can find some p in $|z| \leq 1$ with $M(g) = g(p)$, all g in $[f_1, f_2]$. Hence

$$(26) \quad g(p) = g(x), \text{ all } g \text{ in } [f_1, f_2].$$

But this contradicts (8). Hence (5) must hold.

Thus V , K , P satisfy all the conditions of Lemma 1. That Lemma can therefore be applied, and it yields exactly that P , and so R , is dense in A . Thus Theorem 1 is proved.

COROLLARY OF THEOREM 1: Let ϕ_1, \dots, ϕ_n be complex-valued continuous functions on the unit interval $0 \leq x \leq 1$ which together separate points on the interval. Assume

(i) Each ϕ_i is analytic at each point of $[0, 1]$

(ii) $\phi_1'(x) \neq 0$ for $0 \leq x \leq 1$.

Then every continuous function on $[0, 1]$ can be uniformly approximated there by polynomials in ϕ_1, \dots, ϕ_n .

PROOF: Because of our hypotheses we can find a simple closed analytic curve β in the $z = x + iy$ -plane with the following properties: the unit interval lies in the interior of β and each ϕ_i is analytic inside and on β and $\phi_1' \neq 0$ inside and on β , and the ϕ_i together separate points inside and on β . By a conformal map of the interior of β on the open unit disk we transform the ring R_0 of polynomials in ϕ_1, \dots, ϕ_n into a ring R of functions analytic on the closed unit disk \bar{E} . This ring R then satisfies hypotheses (1) through (3). By a suitable choice of β and using (ii) we also get (4). Hence Theorem 1 applies to R . Hence R is dense in A . Hence R_0 is dense in the corresponding algebra of functions defined on β and its interior. In particular the function z is uniformly approximable there by polynomials in ϕ_1, \dots, ϕ_n . But every continuous function on $[0, 1]$ is a uniform limit on $[0, 1]$ of polynomials in z . Hence every continuous function on $[0, 1]$ can be uniformly approximated on $[0, 1]$ by polynomials in ϕ_1, \dots, ϕ_n . This was the assertion of our Corollary.

NOTE: Without hypotheses (i) and (ii) the conclusion of the Corollary is no longer true. (Cf. [1].)

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SPECTRAL DECOMPOSITION OF OPERATORS IN A BANACH SPACE AND THE ANALYTIC CHARACTER OF THEIR RESOLVENT

Frantisek Wolf

The purpose of this paper is to study linear operators T on a Banach space, whose spectrum $\sigma(T)$ lies on the unit circle C . Results have been established mainly by J. Wermer and the author. J. Wermer [4] studied the existence of invariant subspaces while the author was mainly interested in the possibility of obtaining a generalization of the spectral decomposition theorem. The two problems are closely related. We start from the resolvent $R_\lambda(T) = (\lambda I - T)^{-1}$, which in this case is an operator depending analytically on λ for $|\lambda| \neq 1$. This approach is close to some work by Köthe [2, 3] and Grothendieck [1] who studied the topology of the situation and duality theorems. I propose to review the different connections between these studies and the rather unexpected link to the theory of quasi-analytic periodic functions.

Among the familiar operators closest to our class are the unitary operators. They also have their spectrum on the unit circle. Their spectral decomposition

$$f(T) = \int_C f(e^{i\theta}) E(d\theta)$$

establishes a mapping $f \rightarrow f(T)$, from the Banach algebra B_a of bounded Borel-measurable functions into the bounded operators in Hilbert space. It also establishes a mapping from the Borel-measurable sets S of $\sigma(T)$ to invariant subspaces $M(S)$ in Hilbert space. This mapping is of a local character and -- this is one of the most essential features of the spectral decomposition theorem -- a change of f in S will affect $f(T)$ only in the subspace $M(S)$. My main concern has been to see how general we can let T be and still preserve this local character of the mapping.

The local character depends upon the fact that there are enough "local functions" in B_a : We take $f \in B_a$ to be zero in S and different from zero elsewhere. There exists such an f . Then $M(S)$ will be the null-space of $f(T)$. It can be shown that $M(S)$ depends only on S and not on the choice of f . To every non-empty section of $\sigma(T)$ corresponds a non-zero subspace. If $\tilde{S} = \sigma(T) - S$ contains a non-empty section of $\sigma(T)$, then $M(S)$ is also different from the whole space.

So, if we want to establish such a correspondence between subsets S of $\sigma(T)$ and invariant subspaces for sufficiently many S , then we need a mapping from a Banach algebra of functions to

bounded operators, such that the Banach algebra contains sufficiently many functions. It is especially necessary that it contain "local functions", i. e. functions that are zero outside arbitrarily small intervals. This will furnish a mapping closely paralleling the one for unitary operators.

The first step in generalization is to allow T to be such that its resolvent is $O(1/|1 - |\lambda||^{n-2})$ for a suitable n . The details and proofs of the results mentioned from now on are given in a paper by the author [7].

Examples of such operators are easy to give: The translation and hence also the differentiation operators in $L^2(-\infty, \infty; \sigma)$ are of this nature for suitable measures σ .

It has been proved that for any $f \in C^n$, the Banach space of n times continuously differentiable periodic functions, $f(T)$ can be defined by means of a distribution. This I write for the sake of convenience in a form used by Bochner in the theory of trigonometric integrals: $f(T) = \int f(e^{i\theta}) d^n E(\theta)$ ($d^n E(\theta)$ is essentially $E^{(n)}(\theta) d\theta$, but $E^{(n)}$ need not actually exist). Here $E(\theta)$ turns out to be a continuous operator-valued function of θ and, since $f \in C^n$, the integral is defined by integration by parts. Wherever E is a polynomial of order less than n , there the values of f do not matter.

If we eliminate these intervals from the domain of integration, we obtain $f(T) = \int_{\sigma(T)} f(e^{i\theta}) d^n E(\theta)$. Formally this is a very satisfactory analogue of the spectral decomposition theorem. It is interesting to note other characterizations of this class Γ of operators for which there exists an n so that this last formula holds. The operators in Γ are also exactly those for which there exists an n such that $R_\lambda(T) = O(1/|1 - |\lambda||^n)$. The n 's in the two definitions don't have to be the same. They may differ by anything less than two units. Γ is also the class of operators such that there exists an n for which $||T^m|| = O(|m|^n)$ for $\pm m = 0, 1, 2, \dots$.

The construction of invariant subspaces is almost the same as in the classical case. For any S closed, we can construct an $f(e^{i\theta})$, "completely" vanishing on S , i. e. vanishing with its first $n-1$ derivatives. The null-space of this $f(T)$ is an invariant subspace $M(S)$. It is non-zero under the same conditions as before.

The existence of invariant subspaces has been pushed further by Wermer [4]. His main result is: If there exists a sequence $\{d_n\}$, such that $||T^n|| \leq d_n$, $d_n \geq 1$, d_n non-decreasing, $\log d_n/n$ decreasing, $\sum_{n=1}^{\infty} \log d_n / (1 + n^2) < \infty$, then, if $\sigma(T)$ has at least two points, T has a non-trivial invariant subspace. Similarly the author proved later:

SPECTRAL DECOMPOSITION IN A BANACH SPACE

If $\int_{1/2}^2 \log^+ \log^+ \max_{\phi} \|R_{i\phi}(T)\| dr < \infty$, then there is a correspondence between closed intervals and invariant subspaces. There are functions $f \in B_a$ for which $f(T)$ is a bounded operator and which vanish outside an arbitrary interval.

Let us examine this Banach algebra of functions for which $f(T)$ is a bounded operator. How can we get the maximal Banach algebra for a given operator? If f is analytic on C , then $f(T)$ can be defined in the ordinary way as a bounded operator $f(T) = \int_K R_{\lambda}(T) f(\lambda) d\lambda$, where K denotes two rectifiable curves surrounding $\sigma(T)$ and lying inside the domain of analyticity of f .

Into this algebra let us introduce a norm for f :

$\|f\| = \|f(T)\|$, where $\|f(T)\|$ denotes the operator norm.

This normed algebra is in general not complete. Its completion is the desired Banach algebra which exhibits many properties of the operator T .

If the Banach algebra has sufficiently many functions then we can repeat the reasoning and establish a local mapping $S \rightarrow M(S)$. If this is not so, then there are many different alternatives. For instance, suppose that for a particular point λ_0 no non-zero function of B_a can vanish in the neighborhood of λ_0 . We could say that

the Banach algebra behaves like a class of functions quasi-analytic at one point. Or this could happen at all points of a set. It is obvious that with quasi-analytic Banach algebras, we cannot repeat our construction. This might be an indication that no such mapping $S \rightarrow M(S)$ exists, except in the trivial cases when S is empty or $S = \sigma(T)$.

For an example we may consider an operator about which Wermer [5] had proved that it did not have any invariant subspaces. In the Hilbert space of two-sided sequences $a = \{a_n\}$, with the inner product $(a, b) = \sum_{-\infty}^{\infty} a_n \overline{b_n} p_n$ where $p_n = e^{-|n|/\log |n|}$, take $T\{a_n\} = \{a_{n-1}\}$. If $f(T) = \sum q_n T^n$ represents an operator with bound M , then it can be shown that $|q_n| \leq \sqrt{M} e^{-|n|/2 \log |n|}$. From this it follows easily that the class of functions f for which $f(T)$ is bounded is such that the $f(e^{i\lambda})$ form a quasi-analytic class.

It seems that this is a new construction of quasi-analytic classes. Operators need not have a uniformly singular makeup in different parts of the spectrum. Hence in general, our Banach algebra may be quasi-analytic on one part of the spectrum, analytic on another part and full of local functions in a third. This general situation seems not to have presented itself before. However, there

are classes of functions, studied by Beurling and by Wermer [6], that have the property that any function vanishing near infinity must be identically zero.

Now, an obvious question is: Is there a scalar situation that is similar? Is it possible to generate quasi-analytic Banach algebras by the scalar analogue of the situation described?

Suppose $u(f(z))$ is a linear functional on the space of functions $f(z)$ analytic on the unit circle. Then $u(\frac{1}{\lambda - z})$ is well-defined and analytic in the parameter λ both inside and outside of the unit circle. It is essentially Fantappi 's indicatrix. It in turn determines u uniquely: Indeed, as K the [3] shows $u(f(z)) = \frac{1}{2\pi i} \int_K f(\lambda) u(\frac{1}{\lambda - z}) d\lambda$. If we compare this with the linear mapping $f \rightarrow f(T)$, then we see that $f(T)$ corresponds in the scalar case to $u(f(z))$ and $(\lambda - T)^{-1} = R_\lambda(T)$ corresponds to $u(\frac{1}{\lambda - z})$. Hence, just as the indicatrix determines the linear mapping (or generalized distribution, K the's "Randverteilung"), so $R_\lambda(T)$ determines the spectral mapping $f \rightarrow f(T)$.

The first who seems to have used a pair of analytic functions, determined in complementary domains of the complex plane as representing a generalized distribution was Carleman in his "L'Int grale de Fourier". He defined the Fourier transform of

certain functions, which normally do not have a classical Fourier transform, as a pair of functions analytic in the upper and lower half-planes respectively. There is no doubt that the same idea must be implicit in work of Bochner and Wiener.

The problem of generalized distributions and the related Banach algebras of functions is evidently so rich in ramification in the most unexpected directions that any further work should prove very fruitful.

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