Caro Paolo
spero di venire a Torino
a parlare dei lavori di G. Gherardelli
o di altri argomenti (Varietà
di Buchsbaum, successioni regolari
relative, ecc.)
mandami sempre notizie del tuo
lavoro di ricerca (abbiamo varie
questioni di cui possiamo di-
scutere).

Un abbraccio

Mario Fiorentini

350

Two Proofs of a Theorem Concerning Algebraic Space Curves*

by

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1. Introduction and statement of the result.

It is an obvious truism that mathematics evolves continuously in a non-linear fashion. We shall illustrate this by giving two proofs of a known theorem in algebraic geometry. This result has been discovered independently from completely different viewpoints on at least two occasions, and to some extent our proofs reflect this circumstance.

The result concerns an algebraic curve C in projective 3-space $P^3 = P$. We assume that C does not lie in a plane, and shall refer to $C \subset P$ as a space curve; that these have interested algebraic geometers for over a century is attested to by [B]. When first coming to the subject, one's initial observation is that, in contrast to the single defining equation of an arbitrary plane curve, a space curve generally requires more than two equations to describe it. Equivalently, in general such a curve is not the intersection of two smooth surfaces S and T meeting transversely along C. Curves of the form $C = S \cap T$ are called complete intersections, and the result we shall discuss gives necessary and sufficient conditions that this should be the case.

In addition to C being smooth and irreducible, an obvious necessary condition is that the degree d should factor as

$$d = mn$$

where $m, n > 1$ are the respective degrees of S and T. A further necessary condition is that the canonical divisor K_C should be a multiple of the hyperplane series, or that the equivalent sheaf-theoretic formulation

$$(1.1) \qquad \Omega^1_C \cong \mathcal{O}_C(k)$$

si ottiene partendo dalla succ. di Eulero e applicando la potenza esterna

should hold. Indeed, it follows from $\Omega^3_P \cong \mathcal{O}_P(-4)$ and the respective adjunction formulas for S in P and $S \cap T$ in S that $\Omega^1_C \cong \mathcal{O}_C(m+n-4)$.

A third necessary condition is that $C \subset P$ should be projectively normal. By definition this means that the hypersurfaces of any degree ℓ should cut out on C a complete linear system. Equivalently, for all ℓ the restriction maps

$$H^0(\mathcal{O}_P(\ell)) \longrightarrow H^0(\mathcal{O}_C(\ell))$$

should be surjective. Taking into account the cohomology sequence of

$$0 \to I_C(\ell) \to \mathcal{O}_P(\ell) \to \mathcal{O}_C(\ell) \to 0$$

and fact that

Ext $\qquad H^i(\mathcal{O}_P(\ell)) = 0 \quad$ for $\quad i = 1,2$ and $\ell \in \mathbb{Z}$, projective normality is equivalent to

(1.2) (*) $h^1(I_C(\ell)) = 0$ for all ℓ,

where I_C is the ideal sheaf of the space curve.

We should like to make a couple of observations concerning projective normality. The first is that complete intersections are projectively normal, as follows from the cohomology sequences of

$$1)\begin{cases} 0 \to \mathcal{O}_P(\ell-m) \to \mathcal{O}_P(\ell) \to \mathcal{O}_S(\ell) \to 0 \quad \left(\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^3}/I_S\right) \\ 0 \to \mathcal{O}_S(\ell-n) \to \mathcal{O}_S(\ell) \to \mathcal{O}_C(\ell) \to 0 \end{cases}$$

and $h^1(\mathcal{O}_S(k)) = 0$ for all k. The second is that if we consider the pair of graded rings

$$\begin{cases} R = \text{homogeneous coordinate ring of } C \subset P, \text{ and} \\ \tilde{R} = \bigoplus_{\ell \geq 0} H^0(\mathcal{O}_C(\ell)), \end{cases}$$

then since $h^1(I_C(\ell)) = 0$ for $\ell \geq \ell_0$ it follows that $\tilde{R}$ is a module of finite type over R, while $\tilde{R}$ is easily seen to be integrally closed. Since projective normality is the same as $R = \tilde{R}$, we deduce that this condition is equivalent to the normality, in the usual sense, of the local ring at the vertex of the cone over C. Finally, by localizing at a possibly singular point we see that any projectively normal space curve is first of all smooth, and secondly is irreducible.

The result we shall prove is this:

Theorem: A space curve is a complete intersection if, and only if, conditions (1.1) and (1.2) are satisfied.

As previously mentioned this result is not new. It appears in the paper [G] by Guiseppe Gherardelli in 1936 with a proof along classical lines, and some 25 years later an independent and completely different approach was given by Serre [S]. The two proofs we shall give are roughly parallel to these. The first uses a counting argument and the Riemann-Roch theorem for the curve to estimate the shape of the graph of $h^0(\mathcal{O}_C(\ell))$ as a function of ℓ, while the second uses the interplay between codimension-two subvarieties and rank-two vector bundles, a topic of current interest which was in fact initiated in [S]. The purpose of this paper is expository; it was our goal to illustrate how one arrives at a nice little theorem in geometry by either classical or modern techniques, and to offer these two approaches for comparison.

The notations and background material are standard, and may be found in [G-H].

2. First proof of the theorem.

We recall that C has degree d and that $\Omega_C^1 = \mathcal{O}_C(k)$. Let m be the smallest number such that C lies on a surface of degree m, and let S be such a surface; let $n \geq m$ be the smallest number such that C lies on a surface of degree n not containing S, and let T be such a surface. Clearly

$$d \leqq mn, \tag{2.1}$$

and equality holds if, and only if, $C = S \cap T$ is a complete intersection.

The idea of the proof is this: By the assumption of projective normality, we know the values $h^0(\mathcal{O}_C(\ell))$ for $\ell \leqq n-1$. On the other hand, using duality

$$h^0(\mathcal{O}_C(k-\ell)) = h^1(\mathcal{O}_C(\ell))$$

and the Riemann-Roch gives

$$h^0(\mathcal{O}_C(\ell)) - h^0(\mathcal{O}_C(k-\ell)) = d\ell-g+1 . \tag{2.2}$$

We see from (2.2) that the second differences of the sequence of numbers $h^0(\mathcal{O}_C(\ell))$ has a symmetry, which when combined with the knowledge of $h^0(\mathcal{O}_C(\ell))$ for $\ell \leqq n-1$ and (2.1) will yield the theorem.

Coming to specifics, we set

$$\begin{cases} \alpha_\ell = h^0(\mathcal{O}_C(\ell)) - h^0(\mathcal{O}_C(\ell-1)) \\ \beta_\ell = \alpha_\ell - \alpha_{\ell-1} \\ \qquad = h^0(\mathcal{O}_C(\ell)) - 2h^0(\mathcal{O}_C(\ell-1)) + h^0(\mathcal{O}_C(\ell-2)) . \end{cases}$$

By the assumption of projective normality, for $-1 \leq \ell \leq m-1$

$$h^0(\mathcal{O}_C(\ell)) = h^0(\mathcal{O}_P(\ell)) = \binom{\ell+3}{3} ,$$

so that in this range

$$(2.3) \qquad \alpha_\ell = \frac{(\ell+1)(\ell+2)}{2} , \quad \beta_\ell = \ell+1 .$$

Next, for $m \leq \ell \leq n-1$

$$h^0(\mathcal{O}_C(\ell)) = h^0(\mathcal{O}_P(\ell)) - h^0(\mathcal{O}_P(\ell-m))$$
$$= \binom{\ell+3}{3} - \binom{\ell-m+3}{3}$$

which gives

$$(2.4) \qquad \alpha_\ell = \ell m - \frac{m(m-3)}{2} , \quad \beta_\ell = m .$$

Finally, from (2.2) we have

$$\begin{cases} h^0(\mathcal{O}_C(\ell)) - h^0(\mathcal{O}_C(k-\ell)) = d\ell-g+1 \\ h^0(\mathcal{O}_C(\ell-1)) - h^0(\mathcal{O}_C(k-\ell+1)) = d(\ell-1)-g+1 , \end{cases}$$

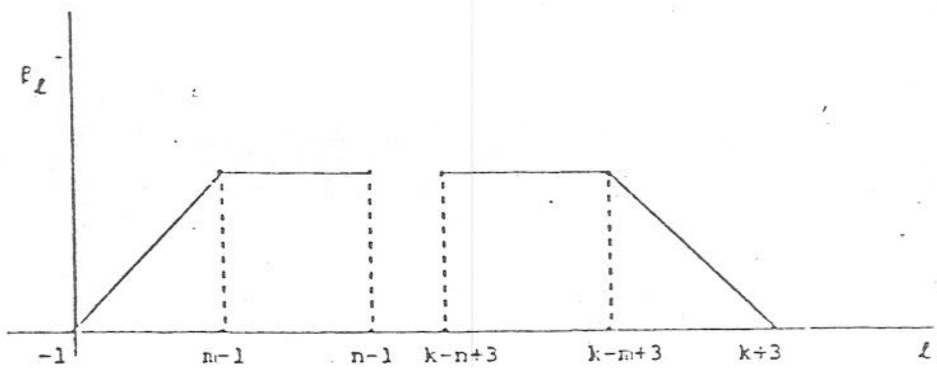
and subtracting gives

$$\alpha_\ell = d - \alpha_{k-\ell+1} ;$$

i.e. from (2.3) and (2.4)

$$\beta_\ell = \beta_{k-\ell+2} = \begin{cases} m & \text{if} \quad k-n+3 \le \ell \le k-m+3 \\ k-\ell+2 & \text{if} \quad k-m+3 \le \ell \le k+2 \\ 0 & \text{if} \quad k+3 \le \ell \end{cases}$$

It follows from (2.3)-(2.5) that the graph of β_ℓ as a function of ℓ has the form.



In particular we see that $k-m+3 \geq n-1$. On the other hand, by (2.1)

$$
\begin{aligned}
mn \geq d = \alpha_{k+3} & \\
&= \sum_{i=0}^{k+2} \beta_i \\
&= \text{area under graph} \\
&= (k+4)m - m^2 \\
&\geq mn
\end{aligned}
$$

since $k+4 \geq m+n$. We conclude that $d = mn$ and $C = S \cap T$ as desired. Q.E.D.

As an immediate consequence of this result about curves, we have

_If_ $V \subset P^r$ _is any variety of codimension_ 2 _such that_

$$H^i(P^r, I_V(k)) = 0 \quad \text{for} \quad 1 \leq i \leq r-2, \text{ all } k$$

and

$$\Omega_V^{r-2} = \mathcal{O}_V(k) \quad \text{for some } k,$$

then V _is a complete intersection._

To see this, let $C = \mathbb{P}^3 \cap V$ be a generic 3-plane section of V. Then

i) By successive applications of the adjunction formula,

$$\Omega_C^1 = \Omega_V^{r-2}(r-3)\big|_C$$

is a multiple of the hyperplane bundle;

ii) Letting $V^{(j)}$ denote an j-fold hyperplane section of V, from the sequences

$$(1.3) \qquad 0 \longrightarrow I_{V^{(j)}}(k-1) \longrightarrow I_{V^{(j)}}(k) \longrightarrow I_{V^{(j+1)}}(k) \longrightarrow 0$$

we see that

$$H^i(\mathbb{P}^{r-j}, I_{V^{(j)}}(k)) = 0 \quad \text{for} \quad i \le i \le r-2-j \text{ , all } k$$

$$\Longrightarrow H^i(\mathbb{P}^{r-j-1}, I_{V^{(j+1)}}(k)) = 0 \quad \text{for} \quad 1 \le i \le r-3-j \text{ , all } k$$

so finally

$$H^1(\mathbb{P}^3, I_C(k)) = 0 \quad \forall k,$$

and C is projectively normal; and

iii) Again from the sequences (1.3),

$$H^0(\mathbb{P}^{r-j}, I_{V^{(j)}}(k)) \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathbb{P}^{r-j-1}, I_{V^{(j+1)}}(k)) \text{ ,}$$

i.e. if $S \subset \mathbb{P}^3$ is any surface containing C, there exists a hypersurface $\overline{S} \subset \mathbb{P}^r$ containing V such that $\overline{S} \cap \mathbb{P}^3 = S$.

Now, from i) and ii) it follows that C is the complete intersection of two surfaces S and T; by iii), then, there exists hypersurfaces $\overline{S}$, $\overline{T}$ in $\mathbb{P}^3$ containing V, with $\overline{S} \cdot \mathbb{P}^3 = S$ and $\overline{T} \cdot \mathbb{P}^3 = T$. Since $\overline{S} \cap \overline{T} \cap \mathbb{P}^3$ is one-dimensional, $\overline{S} \cdot \overline{T}$ must have codimension 2; and since V has degree

$$\begin{aligned} \deg V = \deg C &= \deg S \cdot \deg T \\ &= \deg \overline{S} \cdot \deg \overline{T} \end{aligned}$$

it follows that $V = \overline{S} \cap \overline{T}$.

3. Second proof of the theorem.

The argument proceeds in several steps. The idea is: (i) using (1.1), to associate to our space curve a rank-two bundle $E \to P$ together with a section $s \in H^0(\mathcal{O}_P(E))$ which defines C; and (ii) using (1.2), to show that E is decomposable.

Step one. We consider the data consisting of a holomorphic line bundle $L \to V$ over a smooth threefold and smooth curve $C \subset V$, and ask when there is a pair (E,s) consisting of a rank-two holomorphic vector bundle $E \to V$ and section $s \in H^0(\mathcal{O}_V(E))$ with divisor (s) such that

$$\begin{cases} \det E = L \\ (s) = C \end{cases}.$$

Now, if (E,s) exists then the normal bundle $N_{C/V}$ of C in V is $E|_C$, and from $0 \to T_C \to T_V|_C \to N_{C/V} \to 0$ we deduce the adjunction formula

$$K_C = L \otimes K_V|_C \tag{3.1}$$

We will prove that:

If (3.1) is satisfied, and if $h^2(\mathcal{O}_V(L^*)) = 0$, then (E,s) exists. If, moreover, $h^1(\mathcal{O}_V(L^*)) = 0$ then E is unique.

Proof: We will use standard material on duality and extensions, which can be found, e.g., in Chapter V of [G-H] whose notation we shall use. Assume first that (E,s) exists and write $\mathcal{L} = \mathcal{O}_V(L)$, $\mathcal{E} = \mathcal{O}_V(E)$. Since we are in the rank-two case, the Koszul resolution is the short exact sequence

$$(3.2) \qquad 0 \to \mathcal{L}^* \to \mathcal{E}^* \xrightarrow{s} I_C \to 0$$

where I_C is the ideal sheaf of C in V. The exact sequence (3.2) defines an element

$$e \in \mathrm{Ext}^1(V;\, I_C, \mathcal{L}^*)$$

having the following localization property: For each point $x \in C$, the class e induces

$$(3.3) \qquad e_x \in \mathrm{Ext}^1_{\mathcal{O}_{V,x}}(I_{C,x}, \mathcal{L}^*_x)\ .$$

Since $C \subset V$ is smooth, the right hand side of (3.3) is non-canonically isomorphic to $\mathcal{O}_{C,x}$, but any two isomorphisms differ by a unit. The localization property is that e_x should be a unit.

Conversely, given $e \in \mathrm{Ext}^1(V; I_C, \mathcal{L}^*)$ we have a short exact sequence (3.2) where $\mathcal{E}^*$ is a coherent sheaf. If free of rank two, and we have our desired pair (E,s).

The long exact sequence of Ext's associated to

$$0 \to I_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0$$

gives

(3.4) $$\cdots \to \mathrm{Ext}^1(V;\mathcal{O}_V,\mathcal{L}^*) \to \mathrm{Ext}^1(V;I_C,\mathcal{L}^*) \to$$
$$\to \mathrm{Ext}^2(V;\mathcal{O}_C,\mathcal{L}^*) \to \mathrm{Ext}^2(V;\mathcal{O}_V,\mathcal{L}^*) \to \cdots$$

Recall that we are assuming (3.1), which gives

$$\mathcal{O}_C \otimes \mathcal{L} \otimes \Omega_V^3 \cong \Omega_C^1 ,$$

the tensor product being over $\mathcal{O}_V$. Using this together with

$$\begin{cases} \mathrm{Ext}^q(V;\mathcal{O}_V,\mathcal{L}^*) \cong H^q(\mathcal{L}^*) & \text{, and} \\ \mathrm{Ext}^q(V;\mathcal{O}_C,\mathcal{L}^*)^* \cong H^{3-q}(\mathcal{O}_C\otimes\mathcal{L}\otimes\Omega_V^3) & , \\ \qquad\qquad\qquad\qquad \cong H^{3-q}(\Omega_C^1) & , \end{cases}$$

the second step being duality, the sequence dual to (3.4) is

$$(3.5) \quad \cdots \leftarrow H^1(L^*)^* \leftarrow \mathrm{Ext}^1(V; I_C, L^*) \leftarrow H^1(\Omega_C^1) \leftarrow H^2(L^*)^* \leftarrow \cdots$$

The fundamental class of the curve is a canonical generator for $H^1(\Omega_C^1)$. If $h^2(L^*) = 0$ we deduce from (3.4) and (3.5) the existence of a class $e \in \mathrm{Ext}^1(V; I_C, L^*)$ mapping onto a generator of $\mathrm{Ext}^2(V; \mathcal{O}_C, L^*)$. Since $\mathcal{E}xt^q_{\mathcal{O}_V}(\mathcal{O}_C, L^*) = 0$ for $q < 2$,

$$\mathrm{Ext}^2(V; \mathcal{O}_C, L^*) \cong H^0(\mathcal{E}xt^2_{\mathcal{O}_V}(\mathcal{O}_C, L^*))$$
$$\cong H^0(\mathcal{O}_C)$$

by (3.1), and we deduce that e has non-zero localizations in the sense explained above. Also, e is clearly unique if $h^1(L^*) = 0$.

Step two. We now specialize to a space curve $C \subset P$ of degree d for which (1.1) is satisfied. It follows that $kd = 2g-2$. If we compare with (3.1), use $\Omega_P^3 \cong \mathcal{O}_P(-4)$, and let

$$\begin{cases} L = \mathcal{O}_P(\ell_0) & \text{where} \\ \ell_0 = \dfrac{2g-2}{d} + 4 \end{cases}$$

then from step one and

(3.6) $$h^q(\mathcal{O}_P(\ell)) = 0 \qquad q = 1,2 \text{ and } \ell \in \mathbb{Z} .$$

it follows that there is a unique rank-two vector bundle

$$E \to P$$

together with a section $s \in H^0(E)$ such that

$$\begin{cases} \det E = \mathcal{O}_P(\ell_0) \\ (s) = C . \end{cases}$$

Now a locally free sheaf E on any projective space is said to be <u>decomposable</u> if

$$E \cong \bigoplus \mathcal{O}(k_i) .$$

Specializing to our rank two bundle over P^3, we observe that

<u>E is decomposable if, and only if, the curve is a complete intersection.</u>

Indeed, if $E \cong \mathcal{O}(m) \oplus \mathcal{O}(n)$, then the two direct sum factors of the section $s \in H^0(E)$ define surfaces S and T whose intersection is C. Conversely, if $C = S \cap T$ then obviously C is defined by a section of the rank two bundle $\mathcal{O}(m) \oplus \mathcal{O}(n)$, and by uniqueness we conclude that E is decomposable.

Step three. Suppose now that our space curve satisfies (1.1) and (1.2). By the second step we must show that the vector bundle $E \to P$ is decomposable. From (3.2) and the exact cohomology sequence of

$$0 \to L^*(\ell) \to E^*(\ell) \to I_C(\ell) \to 0$$

together with (3.6), we deduce that

$$h^1(E^*(\ell)) = 0$$

for all ℓ. Using $E \cong E^* \otimes \det E \cong E^*(\ell_0)$ and duality, this condition is equivalent to

(3.7) $$h^q(E(\ell)) = 0 \quad q = 1,2 \text{ and } \ell \in \mathbb{Z}$$

It is a theorem of Horrocks [H] that under the condition (3.7) the vector bundle is decomposable (Horrocks' theorem is for bundles of any rank over arbitrary projective spaces). We shall give a proof of his theorem in our case.

For this recall that, for any line $L \subset P$ the restriction $E_L = E \otimes \mathcal{O}_L$ of E to the line is uniquely decomposable

(3.8) $$E_L \cong \mathcal{O}(m) \oplus \mathcal{O}(n).$$

In general, the integers m and n depend on the line and the bundle is said to be uniform in case they are the same for all lines. It is a theorem of Van de Ven [V] that

A uniform vector bundle is decomposable.

Here is his argument. It will suffice to show that there is an extension

$$0 \to S \to E \to Q \to 0 \tag{3.9}$$

where S and Q are line bundles. Indeed, it follows first that $S \cong \mathcal{O}(m)$ and $Q \cong \mathcal{O}(n)$, and then the extension splits by (3.6). Fix a point $p \in P^3$ and consider the P^2_p of lines passing through p. On any such line we have a unique decomposition (3.6). Suppose first that $m < n$. Then for any point $q \neq p$ we may define the subspace $S_q \subset E_q$ by taking the decomposition (3.8) on the line $\overline{pq}$ and letting S_q be the fibre of $\mathcal{O}(m)$ at that point. This is possible since any automorphism of $\mathcal{O}(m) \oplus \mathcal{O}(n)$ fixes $\mathcal{O}(m)$. It remains to uniquely determine the subspace $S_p \subset E_p$. Letting P^1 be the lines in $E_p \cong \mathbb{C}^2$, we may define a holomorphic map

$$f: P^2_p \to P^1$$

by letting $f(L)$ be the fibre of $\mathcal{O}(m)$ at p for the decomposition (3.6). Such a map is constant, since otherwise $f^{-1}(0) \cap f^{-1}(\infty)$ would give a non-empty set of points where f is not defined. This uniquely determines $S_p \subset E_p$ and gives the desired extension (3.9) in case $m < n$. When $m = n$ we may uniquely determine $S_q \subset E_q$ as before by requiring that $\mathcal{O}(m)_p$ should be a fixed line in E_p; this gives the extension (3.9) in this case.

To complete the proof of our main theorem it remains to prove that

<u>If $E \to P$ satisfies (3.7), then E is uniform.</u>

<u>Proof</u>. For any line L we consider a plane P^2 with $L \subset P^2 \subset P^3$. Together with (3.7) the exact cohomology sequences of

$$\begin{cases} 0 \to E_{P^3}(\ell-1) \to E_{P^3}(\ell) \to E_{P^2}(\ell) \to 0 \\ 0 \to E_{P^2}(\ell-1) \to E_{P^2}(\ell) \to E_L(\ell) \to 0 \end{cases}$$

imply first that $h^1(E_{P^2}(\ell)) = 0$ for all ℓ, and then that

$$H^0(E_P(\ell)) \to H^0(E_L(\ell)) \to 0$$

is surjective for all ℓ. As a consequence, for any integer ℓ, $\dim H^0(E_L(\ell))$ is a constant independent of the line L. We shall show that this implies the uniformity of E.

Indeed, by tensoring E with a suitable $\mathcal{O}(\ell)$ we may assume that on a generic line either

$$E_L \cong \mathcal{O} \oplus \mathcal{O}(-d) \qquad (d > 0),$$

or else

$$E_L \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Suppose now that L specializes to a line L_0. Then by upper-semi-continuity $h^0(E_{L_0}) \geq h^0(E_L)$, and in the first case we can only have $E_{L_0} \cong \mathcal{O}(e) \oplus \mathcal{O}(-d-e)$ $(e \geq 0)$. If $e > 0$, then $h^0(E_{L_0}) > h^0(E_L)$. Similarly, in the second case we can only have $E_{L_0} \cong \mathcal{O}(e-1) \oplus \mathcal{O}(-e-1)$ $(e \geq 0)$, and if $e > 0$ then $h^0(E_{L_0}) > h^0(E_L)$. We conclude that E is uniform, and hence that C is a complete intersection. Q.E.D.

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