

2

Theoretical Physics Introduction

A Single Coupling Constant

The gravitational N -body problem can be defined as the challenge to understand the motion of N point masses, acted upon by their mutual gravitational forces (Eq.[1.1]). From the physical point of view a fundamental feature of these equations is the presence of only one *coupling constant*: the constant of gravitation, $G = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ sec}^{-2}$ (see Seife 2000 for recent measurements). It is even possible to remove this altogether by making a choice of units in which $G = 1$. Matters would be more complicated if there existed some length scale at which the gravitational interaction departed from the inverse square dependence on distance. Despite continuing efforts, no such behaviour has been found (Schwarzschild 2000).

The fact that a self-gravitating system of point masses is governed by a law with only one coupling constant (or none, after scaling) has important consequences. In contrast to most macroscopic systems, there is no decoupling of scales. We do not have at our disposal separate dials that can be set in order to study the behaviour of local and global aspects separately. As a consequence, the only real freedom we have, when modeling a self-gravitating system of point masses, is our choice of the value of the dimensionless number N , the number of particles in the system.

As we will see, the value of N determines a large number of seemingly independent characteristics of the system: its granularity and thereby its speed of internal heat transport and evolution; the size of the central region of highest density after the system settles down in an asymptotic state; the nature of the oscillations that may occur in this central region; and to a surprisingly weak extent the rate of exponential divergence of nearby trajectories in the system. The significance of the value of N is underlined by the fact that N often makes its appearance in the very

names of the problems we study, e.g. the famous “three-body problem”, and in the titles of books like this one.

$$N = 2, 3$$

The three-body problem is so famous precisely because it is one of the oldest problems that *cannot* be solved. In contrast, the two-body problem is one of the oldest *solved* problems. It was Newton’s great triumph that he was able to explain why planets move in elliptical orbits around the sun, as had been discovered earlier by the brilliant Kepler, using observational data by Tycho Brahe. The significance of Newton’s solution can hardly be overestimated, since the result was a breaching of the seemingly separated realms of the temporal, human, sub-lunar world and the eternal world beyond the Moon’s orb.

Though Newton had shown how to solve the two-body problem, the presence of even an infinitesimal third body (the so-called “restricted” three-body problem) is insoluble in the usual sense.* We should not forget, however, that Newton was able to solve some three-body problems approximately. He knew, for instance, that it is better to treat a planet as revolving around the barycentre of the inner solar system than about the sun itself. If we do not insist on a precise solution, as with the two-body problem, in other words, if we look at it from the point of view of *physics* rather than mathematics, as Newton did, many aspects of the behaviour of the three-body problem are not so hard to understand. Indeed it is one of the quiet triumphs of recent decades that our intuitive understanding of three-body motions has developed to the extent it has. Most physics undergraduates are still exposed to the two-body problem, and can easily develop a feel for the motion in ellipses and hyperbolae, especially when the scattering problems of atomic physics are encountered. We think that it is possible to arrive at a similar feel for the richer behaviour of three-body systems, and one of our aims in writing this book is to demonstrate this.

The *mathematical* development of the three-body problem continued in the hands of Euler, Lagrange, Laplace and many others, who systematised the methods of approximate solution. These led to the remarkable and successful development of the theory of planetary motion; the successes of the recovery of Ceres and the discovery of Neptune; and the familiar but no less remarkable ability of astronomers to predict the mo-

* As an aside, it is amusing to see the progress of physics reflected in our (in)ability to solve N -body problems. In Newtonian gravity we cannot solve the 3-body problem. In general relativity we cannot solve the 2-body problem. In quantum electrodynamics we cannot solve the 1-body problem. And in quantum chromodynamics we cannot even solve the 0-body problem, the vacuum.

tions of the planets, satellites, asteroids, and comets. Equally, they led to profound insights in the emerging fields of dynamical systems and chaos (Chapter 4), which in turn have illuminated our understanding of N -body problems. But they also led away from the sort of approach we need if our goal is to find quantitative answers to questions about the million-body problem.

$$N \rightarrow \infty$$

Throughout classical physics there are several important analogues to the million-body problem, and it is from this background, and not so much from celestial mechanics, that the most fruitful approaches have come. At first sight the closest analogy is with plasma physics, where inverse square laws, as in Eq.(1.1), also arise, but the analogy has not proved as fruitful as one might think.

In the first place plasmas are often nearly neutral, and for this reason it makes sense to conceive of plasmas which are nearly uniform, nearly at rest and of large spatial extent. By contrast, it is impossible to conceive of an infinite uniform gravitational medium in equilibrium (Fig.1). Newton himself glimpsed this problem, and solved it by noting that the stars had been placed at immense distances from each other, lest they should, “by their gravity, fall on each other”. This was a satisfactory solution if one supposed that the system of fixed stars were young enough. For stellar systems the modern solution is that the random motions or circulation of the stars maintain *dynamic* equilibrium, just as the motions of the planets prevent their falling into the sun. Even so, we shall see that the contraction of stellar systems in dynamic *quasi*-equilibrium is a real and fascinating issue, though it is a much gentler process than the headlong rush suggested by the phrase “gravitational collapse”. In problems of cosmology, Newton’s dilemma is avoided by an overall expansion or contraction of the entire universe, though a general relativistic treatment is required.

The second basic feature of an infinite plasma is that several important effects are localised within a *Debye length*, beyond which the rearrangement of charges effectively screens off the influence of an individual charge (see, for example, Sturrock 1994). If one naively computes the Debye length for a stellar system, it is found usually to be of the order of the size of the system itself (Problem 2). Thus it is difficult to treat gravitational interactions as being localised within a stellar system, though it is a difficulty that stellar dynamicists habitually ignore. In any event, all these considerations help to explain why many of the well known properties of plasmas have little relevance to stellar systems. And yet there is

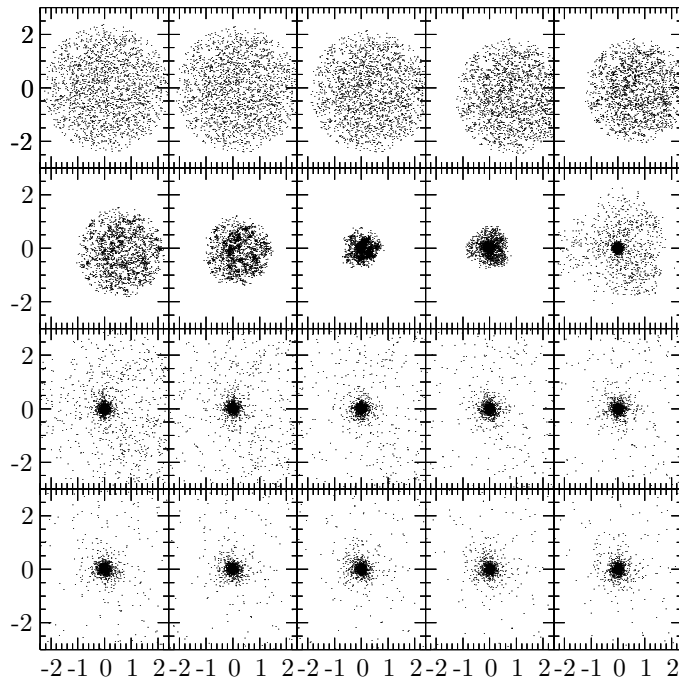


Fig. 1. Collapse of a uniform “cold” sphere. Initially 2048 particles are distributed randomly within a sphere. In units such that $G = 1$, the initial radius is 2.4, the total mass is 1, and successive frames are taken at intervals of $2\sqrt{2}/5$ (cf. Box 1.1). Late in the collapse (frame 7 and 8) the distribution of particles has become very irregular. In frame 9 the particles which arrived first have begun to reexpand. By the last row of frames the remaining central condensation has settled nearly into dynamical equilibrium.

one phrase that stellar dynamicists use all the time which betrays their debt to plasma physics: the words “Coulomb logarithm”, which occur in the theory of relaxation (Chapter 14).

Thermodynamics

The plasma analogy exploits the form of the force equation but not the size of N . Physicists routinely study problems which great numbers of particles using concepts of thermodynamics. How fruitful is such an approach to the million-body problem?

From a formal point of view, unfortunately, the field of thermodynamics excludes a description of self-gravitating systems. The reason is that the existence of gravitational long-range forces violates the notion of an asymptotic thermodynamic limit in which physical quantities are either intensive or extensive. Gravity exhibits what is known in particle physics

as an infra-red divergence. This means that the effect of long-distance interactions cannot be neglected, even though gravitational forces fall off with the inverse square of the distance (Problem 3).

Take a large box containing a homogeneous swarm of stars. Now enlarge the box, keeping the density and temperature of the star distribution constant. The total mass M of the stars will then scale with the size R of the box as $M \propto R^3$, and the total kinetic energy E_{kin} will simply scale with the mass: $E_{kin} \propto M$. The total potential energy E_{pot} , however, will grow faster: $E_{pot} \propto M^2/R \propto M^{5/3}$. Unlike intensive thermodynamic variables that stay constant when we enlarge the system, and unlike extensive variables that grow linearly with the mass of the system, E_{pot} is a superextensive variable, growing faster than linear.

As a consequence, the specific gravitational potential energy of the system, the total potential energy of the system divided by the particle number N , grows without bounds when we increase N . This causes various problems. For example, the kinetic energy of a stable self-gravitating system is directly related to the gravitational potential energy through the so-called virial theorem (Chapter 9). Therefore, we have to make a choice when enlarging a self-gravitating system. Either we increase the temperature steadily while increasing N , in order to increase the specific kinetic energy enough to satisfy the virial theorem and guarantee stability. Or we keep the temperature constant, and quickly lose stability when enlarging our system. In the latter case, the system will ‘curdle’: it will fall apart in more and more subclumps, and the original homogeneity will be lost quickly.

In conclusion, there is no way that we can reach an asymptotic thermodynamic limit, with the system size becoming arbitrarily large while holding the intensive variables fixed*. Therefore, the traditional road to equilibrium thermodynamics is blocked. There are no arbitrarily large homogeneous distributions of stars. As soon as the Universe became old and cold enough to let matter condense out of the original fire ball into ‘islands’ in the form of galaxy clusters and galaxies, the original homogeneity was lost. And each individual clump of self-gravitating material, be it a galaxy or a star cluster, is ultimately unstable against evaporation, and will fall apart into a bunch of escaping particles (Fig.2). Most of these escapers will be single, some will escape as stable pairs, and a few will even manage to form stable triples or higher-number multiples of particles.

* A consistent theory is possible if we let $R \rightarrow \infty$ with N/R fixed (de Vega & Sanchez 2000), though what relevance this might have to the million-body problem is unclear

The presence of these few-body systems is a robust feature of N -body systems, as we shall see throughout this book. From our present perspective, it is an indication of another problem: a short-range (“ultra-violet”) divergence. The usual Boltzmann factor used in calculations of canonical ensemble averages, i.e. $\exp(-E/kT)$, gives divergent results in the limit when two particles approach each other within a small distance r . This factor then contains a term $\exp\left(\frac{Gm^2}{rkT}\right)$.

A Lack of Handles

Even though we cannot use thermodynamics in a formal way, when dealing with a star cluster, we can still describe the motion of the stars in a way that is analogous to the treatment of the motion of molecules in a gas studied in a laboratory. One important difference is that a swarm of stars forms an open system, while a body of gas in a lab has to be contained. Typical textbook experiments in thermodynamics show the gas to reside inside a cylinder, with a movable piston that allows the experimenter to change the volume of the gas. In a star cluster, there are no cylinder and piston. Instead, the stars are confined by their collective gravitational field.

The structural simplicity of a star cluster thus allows far less experimentation than is the case for a body of gas in a lab situation. Whether in

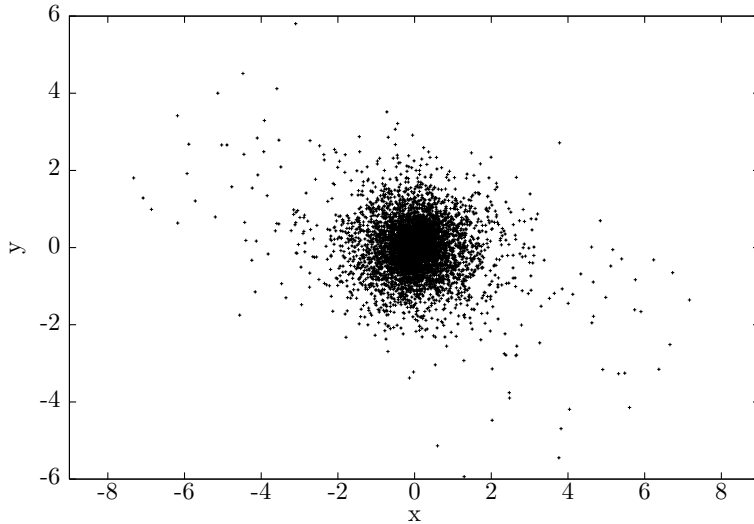


Fig. 2. Escape from an N -body system. This computer model of a star cluster shows a thin stream of escapers emerging at the left and right. The escapers are channeled into these streams because of external forces, and the streams are curved because of Coriolis forces.

thought experiments, computer simulations, or in actual table top experiments, the macroscopic parameters of a laboratory gas can be changed freely, independent of the microscopic parameters governing the attraction and repulsion between individual molecules. Temperature, density, and size of the system all can be varied at will. In contrast, once the number of particles in a self-gravitating system has been chosen, we are left with no degree of freedom at all, apart from trivial scalings in the choice of unit of length, time, and mass.

The fact that there are no dials that can be turned in a self-gravitating experiment, apart from the choice of the total number of stars, is directly related to the ultra-violet and infra-red divergences of classical gravity. Having a simple shape for the gravitational potential energy well, with an energy inversely proportional to distance, leaves no room for preferred length scales. In contrast, molecular interactions show far more complicated forces, typically strongly repulsive at shorter distances and weakly attractive at larger separations between the molecules. This change in behaviour automatically specifies particular length scales, for example the distance at which repulsion changes into attraction. In contrast, gravity is attractive everywhere, at least in the classical Newtonian approximation.

Towards an understanding of the million-body problem

It is now being realised that the gravitational N -body problem is just one of a growing list of known systems with long-range interactions where the non-extensivity of energy looks like an obstacle (Cipriani & Pettini 2001). (In other contexts these are known as “small systems”, to indicate that their spatial extent is comparable with the range of the relevant interaction.) It turns out that non-extensivity is only an issue if we insist on treating these problems with the traditional tools of canonical ensembles (corresponding to systems immersed in a heat bath). This hardly seems natural in the stellar dynamical context. It is the *microcanonical* ensemble which corresponds best to the isolated N -body systems which have been studied so much in stellar dynamics. Indeed it is known that the different ensembles one studies in thermodynamics, and which are usually regarded as equivalent for many purposes, have very different properties for self-gravitating systems (see, for example, Youngkins & Miller 2000).

The *formal* inability to apply traditional thermodynamic concepts, then, does not seriously hinder us from thinking and working with them. Gravity is just as important in the theory of stellar structure and evolution, where thermodynamics is just as much the stock-in-trade as in any other area of gas dynamics. Indeed it was precisely the search for extrema of the entropy of stellar systems that led V.A. Antonov (Antonov 1962)

to one of the most profound insights into the behaviour of stellar systems – gravothermal stability.

In addition to fundamental approaches like this, there are several other routes by which the behaviour of N -body systems are explored (Chapter 9). One can construct toy models, one can borrow models from other areas (such as the theory of stellar evolution, or the kinetic theory of gases), and these can be studied analytically or by numerical methods. Finally, simulations may be based, with the minimum of simplifying assumptions, on the numerical integration of Eqs.(1.1). As a result of all these types of approaches, over the last few decades we have developed a relatively deep and accurate understanding of the many subtle aspects of the evolution of a system of gravitating point masses, the topic of this book. Indeed the study of the gravitational N -body problem must rank as one of those mature areas of research where the variety of approaches enrich each other like a community of craftsmen.

Let us highlight a few points of interest for a general physics point of view.

Perhaps the most fascinating aspect of self-gravitating systems is their instability*, exemplified by the fact that these systems have a negative heat capacity (Chapter 5). Thermodynamic purists would shudder, but the idea is intuitively simple to grasp and very powerful. Removing heat from a system means reducing its kinetic energy of random motions. If this is done to a self-gravitating system, its constituents fall towards each other a little, and in doing so actually pick up more kinetic energy than they lost in the first place. When this concept is applied to *part* of a stellar system (as it easily can be in thought experiments) an instability can result. Imagine what might happen to a block of material with a negative specific heat, as heat flows from a hot spot: the more heat it would lose, the hotter it would get, and it would quickly burst into flame.

Next, the N -body problem is a fascinating example of a system for which no useful equilibrium exists. Particles can and do escape (Fig.2). A Maxwellian distribution of velocities would always allow particles of arbitrarily high speed, but in self-gravitating systems there is a finite escape speed. Therefore the notion of “local thermodynamic equilibrium”, which is such a powerful idea in many areas, has limited usefulness. Even if we prevent particles from escaping (in a thought experiment), thermodynamic equilibrium is possible, but may be unstable (because of the negative specific heat). There is certainly no equilibrium in the sense of solutions of Eqs.(1.1) in which all particles are at rest. Even the con-

* We exclude $N = 2$, and certain kinds of stable larger systems, e.g. hierarchical triples (Chapter 25)

cept of *dynamic* equilibrium (if we ignore for a moment the escape of particles) brings with it the difficulty of showing why such equilibria are stable. This in turn depends on an understanding of the global modes of oscillation of a system in dynamic equilibrium, and for stellar systems like globular clusters this area is still in its infancy, despite a great deal of work by many experts. One of the obstacles, of course, is the severe spatial inhomogeneity of gravitating systems.

The N -body problem exemplifies some of the most perplexing issues in statistical mechanics. Even though the equations of motion are reversible, particles escape and are never captured. A stellar system with a collapsing core (Chapter 17) will collapse, even if the velocities are reversed. In every reasonable sense it is a chaotic system, which rapidly forgets its initial conditions, and collisions can almost always be treated with Boltzmann's classic "Stoßzahlansatz". The fact that there are exceptions, e.g. the mutual interaction of stellar orbits in the dominating field of a central black hole (Rauch & Tremaine 1996), is an avenue (so far unexplored) for investigating the foundations and limitations of the stochastic treatment of collisions.

Problems

- 1) How would Newton have solved the problem of the collapse of the system of fixed stars if he had had access to a GRAPE*?
- 2) In plasma physics the Debye length is defined to be

$$\lambda_D = \sqrt{\frac{kT_e}{4\pi e^2 N_e}}.$$

Translate this into the language of stellar dynamics, bearing in mind that the rms speed of electrons is related to the electron temperature by

$$v_{Te} = \sqrt{\frac{3kT_e}{m_e}}.$$

If a stellar system is in dynamic equilibrium then it approximately obeys the *virial relation* $2T + W = 0$, where T is now the *total kinetic energy* and W the total potential energy (Chapter 9). By making suitable estimates for the density and other parameters of the system, show that its radius is comparable with the Debye length.

- 3) Newton's Theorems on the gravitational force due to a uniform spherical shell imply that the force inside vanishes, and the force outside is the same as that due to an equal point mass at the centre of the

* instead of an Apple

shell. Hence show that the force at a point due to an infinite uniform medium can take any value we please.

- 4) Using Newton's Theorems (Problem 3) show that the acceleration of a point at a distance r from the centre of a uniform sphere of finite radius a and total mass M is

$$\ddot{r} = -\frac{GMr}{a^3}.$$

Deduce that the sphere collapses homologously, and that collapse is complete at time

$$t = \sqrt{\frac{\pi^2 a_0^3}{8GM}},$$

where a_0 is the initial value. Compute this for the system displayed in Fig.1.