4
Mathematical Introduction

For the mathematician, the gravitational N-body problem is the problem of understanding *by pure thought* the solutions of the set of differential equations

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1, j\neq i}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \tag{1}$$

where $\mathbf{r}_j$ is the position vector of the jth body at time t, m_j is its mass, G is a constant, and a dot denotes differentiation with respect to t. Superficially, what distinguishes the work of a mathematician from that of, say, an astrophysicist, is its organisation into theorems, lemmas, and so on, but that is simply a matter of style. There are few formal theorems in Poincaré's "Les Méthodes Nouvelles de la Mécanique Céleste", but it is a rich vein of ideas. At a deeper level, the work of the mathematician aims at a greater level of rigour.

Apparently it was Herman (1710; see Volk 1975) who first solved the two-body problem using Eq.(1) (in component form). Since then this manner of expressing the problem has proved remarkably resilient: much the same form of equation for the general case can be found over 200 years later in the book by Moser (1973). Though Eqs.(1) are usually referred to as "Newtonian", there is nothing like them in any of Newton's published works or writings. Instead, his expositions are dressed in the language of geometry or infinitesimals. Curiously, the modern language of geometry has taken an increasingly important role in recent decades: Box 1 shows a recent statement of the two-body problem, in terms of a manifold M and its canonical symplectic structure. Normally, however, we prefer to work with Eq.(1).

Box 1. *The modern two-body problem.*

In relatively modern language the two-body problem may be defined to be (Abraham & Marsden, 1978) the "system (M, H^μ, m, μ), where:
(i) $M = T^*W$ with canonical symplectic structure,

$$W = \mathbf{R}^3 \times \mathbf{R}^3 \backslash \mathbf{\Delta}, \qquad \mathbf{\Delta} = \left\{(\mathbf{q},\mathbf{q}) | \mathbf{q} \in \mathbf{R}^3\right\};$$

(ii) $m \in M$ (initial conditions);
(iii) $\mu \in \mathbf{R}$, $\mu > 0$ (mass ratio);
(iv) $H^\mu \in F(M)$ given by

$$H^\mu(\mathbf{q},\mathbf{q}',\mathbf{p},\mathbf{p}') = \frac{\|\mathbf{p}\|^2}{2\mu} + \frac{\|\mathbf{p}'\|^2}{2\mu} - \frac{1}{\|\mathbf{q}-\mathbf{q}'\|}$$

where $\mathbf{q},\ \mathbf{q}' \in \mathbf{R}^3$, $\mathbf{p},\ \mathbf{p}' \in (\mathbf{R}^3)^*$ and $\|\ \|$ denotes the Euclidean norm in $\mathbf{R}^3$."

Our opening remark begs the question of what is meant by a "solution". The very *existence* of a solution, at least locally, is assured by the usual undergraduate theorem in a course on ordinary differential equations (e.g. Arnold 1978b). Globally, the obvious pitfalls are the numerous surfaces (in phase space) where singularities of the differential equations occur, i.e. two-body singularities (a "hypersurface" S_{ij} where $\mathbf{r}_i = \mathbf{r}_j$ for some pair i, j), three-body singularities (where two two-body surfaces S_{ij} and S_{ik} intersect), and so on.

Simply by counting dimensions it is easy to see that any given orbit in phase-space is very unlikely to encounter one of these singularities. The argument may be illustrated in a three-dimensional context. A single condition such as $x^2 + y^2 + z^2 = 1$ determines a two-dimensional surface. An orbit is one-dimensional, and if a given orbit intersects the surface then neighbouring orbits do (Fig.1). On the other hand if we have two conditions to satisfy, each yields a surface, and both conditions are satisfied only on their intersection, which is a curve (one-dimensional). It is still possible for the orbit to intersect this curve, but neighbouring orbits do not, in general. This argument shows that curves which intersect two surfaces simultaneously are rare in three dimensions, or "of measure zero".

Returning to the N-body problem, it is easy to show that orbits intersecting any of the surfaces S_{ij} are rare, and so two-body and higher-order collisions are rare in the same sense. The situation changes dramatically, however, if we allow the position vectors $\mathbf{r}_i$ to depend on a *complex* time

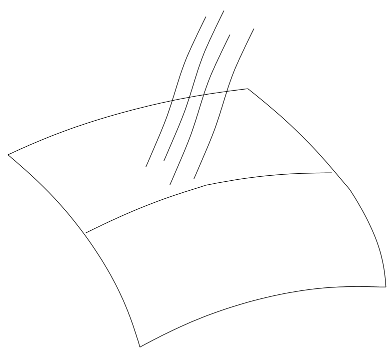

Fig. 1. If an orbit in three dimensions intersects a two-dimensional surface, neighbouring orbits do. If it intersects a one-dimensional curve, neighbouring orbits do not (in general).

variable t (Box 2). Why should one do this? One answer is that it is an important setting in which to discuss the analytical properties of the solutions, not only because of its pure-mathematical significance, but for "practical" reasons also. For example, the numerical treatment of the equations of motion requires special care in the vicinity of singularities (cf. Chapter 22), and in the N-body problem these are usually to be found in the complex t-plane. Another application (Chapter 21) is in certain problems of three-body scattering, when a third body temporarily approaches a short-period binary star; it turns out that the change in its energy is determined by the point in the complex plane where the intruder collides with the binary.

We have treated the two-body singularities as though they are to be avoided at all costs. In fact they are quite innocuous. For example, suppose a collision occurs in a two-body problem when $t = 0$, and we try to study this problem by numerical integration of the equations of motion, starting at some negative value of t during the approach to the collision. As t approaches 0 from below, it will be found that the numerical integration requires ever smaller time steps as we approach the singularity at $t = 0$, and there is nothing that can be done to get past it. Our local theorems about the existence of solutions also take us no further. When we examine the same problem analytically, however, it turns out that the coordinates of the two bodies are expressible, in the run-up to the collision, as power series in the variable $t^{2/3}$ (Box 2). Since this series contains *non-integer* powers of t we recover the fact that x is not an analytic function at $t = 0$. But we also observe that the series can be expressed in powers of $(t^2)^{1/3}$, which may be equally well evaluated when

Box 2. *Singularities of the two-body problem.*

For the planar two-body problem, complex singularities are easily located using the exact solution. A negative-energy solution of the equation

$$\ddot{\mathbf{r}} = -\mathbf{r}/r^3 \tag{1}$$

may be represented as

$$\mathbf{r} = (a(\cos E - e), b\sin E), \tag{2}$$

where a, b and e are constants such that $b = a\sqrt{1-e^2}$, and E is determined by Kepler's equation; this is

$$n(t - t_0) = E - e\sin E, \tag{3}$$

where n and t_0 are more constants, with $n^2a^3 = 1$. From Eq.(2) it follows that $r = a(1 - e\cos E)$, and so a two-body collision occurs where $\cos E = 1/e$, i.e. where $E = \pm i\cosh^{-1}(1/e) + 2\pi m$, in which m is an integer. By Eq.(3), this corresponds to the complex times $t = t_0 + (2\pi m \pm i\cosh^{-1}(1/e) - i\sqrt{e^{-2} - 1})$.

Now let $e = 1$ and $t_0 = 0$ in Eqs.(2) and (3). Then a collision occurs at $t = 0$, and the analytical solution may be written as $\mathbf{r} = (x, 0)$, where

$$x = a(\cos E - 1) \tag{4}$$

and $nt = E - \sin E$. In the vicinity of the collision at $t = 0$ we have $nt = E^3/6 + O(E^5)$ and $x = -aE^2/2 + O(E^4)$. It is not hard to see from the first of these equations that E is expressible as a series in odd powers of $t^{1/3}$, and then the second equation shows that x may be developed as a series in powers of $t^{2/3}$.

t is positive as when t is negative. Furthermore, it turns out that this observation is equally valid when we consider two-body collisions which are perturbed by other bodies, where the argument used in Box 2 (which is based on the exact solution) no longer holds. In any event, it turns out that this approach shows that there is an *analytic continuation* of the solution beyond the two-body singularity.

Some additional steps are needed before this idea can be turned to practical use, and we shall have something to say about this important technique later on (Chapter 15). In the meantime we may say that there is a choice of independent variable (the quantity $t^{2/3}$ in the above discussion) which allows us to represent the solution of the N-body problem as a function which is analytic on the real axis, even if there are two-body collisions. We may say that the collision singularities have been *regularised*.

There is an artificial but mathematically important class of N-body problems in which these techniques are essential. Two-body collisions occur naturally if one studies the *collinear* N-body problem, but when these collisions have been neutralised, other, higher-order singularities come into view. The most obvious of these is the *triple-collision* singularity, in which the coordinates of three bodies tend to coincidence as some value of t is approached. One way of studying this problem is touched on in Chapter 21, and it shows that these singularities cannot usually be handled by the technique of analytic continuation which works so well for two-body collisions. The problem is that the exponents which occur in the corresponding power series are not usually simple rational numbers, such as the power of 2/3 which occurs in the case of two-body encounters. Instead, the exponents may involve irrational numbers such as $\sqrt{13}$ (Chapter 21), even in the simplest case of equal masses, and a series involving such powers of t cannot be evaluated (with a real result) for both positive and negative values of t. Nevertheless the study of these singularities has progressed to remarkable lengths, and these investigations are not without their consequences for the astrophysical applications of the N-body problem.

The collinear N-body problem exhibits also other classes of singularities, in which no more than two bodies collide at one time, but the collisions occur more and more frequently as a certain time approaches (see Marchal 1990). In order to give an impression of how this can happen, we have to anticipate some results of Chapter 21. Consider first the notion of a binary. In the collinear problem this is a pair of stars exhibiting a relative motion like that in Eq.(4) of Box 2: they periodically bounce off each other (as we assume that the relevant two-body encounters have been regularised). Just after one such bounce the right-hand component moves to the right at high speed. Now suppose a third body of low mass approaches from the right and collides with the right-hand component (Fig.2). After this collision the third body recedes at high speed, its energy having been gained at the expense of the binary, which becomes "tighter". Suppose finally there is a fourth body, to the right of the third and moving off to the right. The third body, moving very fast, catches up with it and, being of relatively low mass, bounces back towards the binary. With sufficient care its next encounter can be arranged to occur, once again, just after a collision between the binary components. (In Chapter 20 we shall see in a little more detail how careful choice of initial conditions can lead, in an analogous problem, to an orbit with desired properties.) Now we have the design of a powerful four-body machine which, it may be shown, can accelerate the middle body to arbitrarily high speeds within a finite time.

When we return to a more reasonable number of dimensions it is not hard to avoid triple collisions in the three-body problem. All that is needed is to endow the system with non-zero angular momentum in its barycentric frame. The essential reason is that, if all three particles could be confined (however briefly) into a sphere of radius r, energy conservation shows that their speeds would scale as $r^{-1/2}$ and so their angular momentum would scale as $r^{1/2}$. Thus confinement within an arbitrarily small volume is inconsistent with non-zero angular momentum.

Even though collisions are usually avoided in three dimensions, singularities analogous to the one shown in Fig.2, though without collisions, are still possible, at least for the 5-body problem. The story of how this remarkable result was obtained (by J. Xia) is beautifully told in Diacu & Holmes (1996); cf. also Saari & Xia (1995). Essentially, Xia's example consists of two Sitnikov machines (see Chapter 20) coupled end-to-end, with cunningly contrived initial conditions.

Though these examples might seem like mathematical playthings, they bear some resemblance to a curious idea with possible implications (admittedly, in the very long run) for mankind. As the sun expands and heats up it may be possible (in principle) to keep the Earth cool by making its orbit expand. This is done by repeated two-body encounters involving the Earth and an asteroid, and Jupiter and the asteroid (Korycansky et al. 2001).

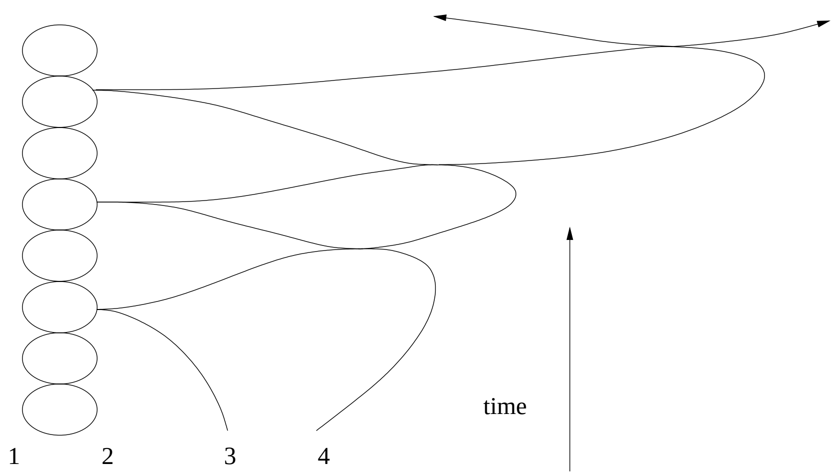

Fig. 2. Design of a four-body machine for accelerating a particle to infinite speed in a finite time.

Examples like Xia's are highly contrived and rare. In the N-body problem there will usually be no singularities on the real time-axis. Even if we regularise two-body collisions, however, there will usually be plenty of singularities in the rest of the complex plane. These prevent us from being able to express the solution of the N-body problem as power series in t (or the appropriate regularising variable) which converge for all times. However there is an amazingly simple transformation of the independent variable (Box 3) which does allow us (in principle) to write the solution as a series which converges for all times; it is not a power series in t, however. Unfortunately the solution in this form has never been put to practical use.

Since a solution of the 3-body problem may be expressed as a convergent series, it is surprising to recall that this problem is often quoted as one of the famous unsolved problems of applied mathematics. Clearly, the issue hinges on what is meant by a "solution". Though the series expression is a solution of sorts, it would be very hard to extract from this series any information about the qualitative behaviour of the motion of the N bodies (or even any quantitative results). Nor is it very useful for numerical calculations as the rate of convergence is even more painfully slow than an N-body simulation. One usually expects much more from a satisfactory solution of a dynamical problem.

The best known class of soluble problems in dynamics are the so-called "integrable" problems. We shall take this to mean a problem in which sufficiently many constants of the motion can be found so that the solution may be written down in terms of "quadratures", i.e. an integral of a function of a single variable. How one finds these integrals is another matter, and usually boils down to identifying a symmetry of the problem at hand. For example the motion of a particle in any spherical potential may be integrated using the fact that the angular momentum and energy are constant, and the existence of these integrals results from the invariance of the potential under rotations and time-translation.

The question now arises whether the N-body problem is of this type. For $N = 2$ the answer is affirmative, and indeed it may be reduced to the problem of motion in a spherical potential (cf. Chapter 7). In fact in this case the quadratures can be carried out analytically. Even for $N = 3$, however, insufficiently many integrals are known, and the search for other integrals has even led to theorems proving their non-existence under certain conditions (Whittaker 1927, Moser 1973). Chapter 20 describes a particular kind of three-body problem where this question can be settled rather directly. The existence of integrals was one of the questions which motivated Poincaré to study the three-body problem and, in the process, Box 3. *A series solution of the three-body problem.*

For some solution of the 3-body problem let us suppose that the singularity which is closest to the real t-axis is at a distance $h > 0$ (i.e. its imaginary part is $\pm ih$). Then the solution of the N-body equations is analytic (without singularities) throughout a strip of width $2h$ (Fig.1). Now the complex transformation

$$t = \frac{2h}{\pi} \log \frac{1+\tau}{1-\tau}$$

maps the unit disk $\tau \leq 1$ into the strip just mentioned. For example, when t is real the transformation is $\tau = \tanh(\pi t/(4h))$, and this varies from $\tau = -1$ to $\tau = 1$ as t increases from $-\infty$ to ∞. Since our solution of the N-body problem is analytic in the stated strip in the t plane, it follows that, if τ is used as independent variable, the solution of the N-body problem has no singularities in the unit disk. Therefore, by elementary theorems in the theory of analytic functions, the solution can be expanded as a power series in τ for all τ in the unit disk. In principle, therefore, a solution of the N-body problem may be represented as a convergent power series for all time. See Saari (1990), Barrow-Green (1996).

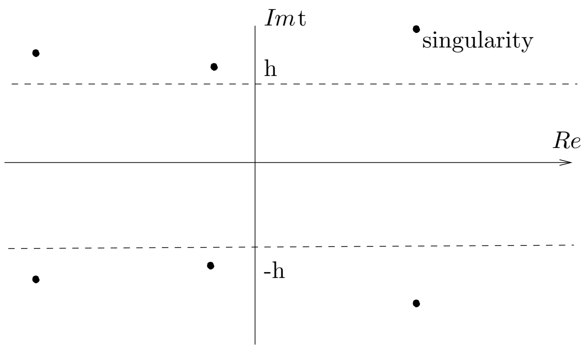

Fig. 1. A solution of the 3-body problem is analytic in a strip $|Im\ t| < h$, which maps to the unit disc in the complex τ plane.

to uncover many of the foundations of current research in Hamiltonian dynamical systems.

What one can do with the known integrals, however, is to reduce the order of the problem, i.e. the dimensionality of the phase space. The integrals associated with the motion of the centre of mass (or barycentre), for instance, reduce the order by 6, but this is not much compared with

$6N$. With all such tricks even the three-body problem can be reduced only to order 7 (though one more order can be removed by transformation of the independent variable). Another way of expressing this is to say that a three-body system moves on a 7-dimensional subspace of phase space. By studying the topology of this subspace one can sometimes reach rigorous conclusions on the stability of three-body systems (the "c^2h" criterion, cf. Marchal 1990). But this topological problem also has a pure-mathematical life of its own which recently led to a complete census of all possible types of topology in this context (cf. Diacu 2000).

The topics raised in this chapter, and even the title of Poincaré's book, illustrate the remarkably close links that have existed between mathematics and dynamical astronomy ever since the time of Newton. It is significant, however, that most of this has occurred within the subject that is even now termed "celestial mechanics". Loosely speaking, this is the mathematical study of few-body problems, usually with one dominant mass, such as those found within the solar system. We think that there is equally fertile ground for such cross-fertilisation between mathematics and the N-body problem of *stellar dynamics*, which is the subject of this book. That this is less well-developed than in the area of celestial mechanics can be traced to the fact that astronomy as a whole has become largely a part of physics. And yet it will be found from certain sections of this book that tools which have been developed by mathematicians for their own inscrutable reasons have turned out to have important applications in stellar dynamics, often several decades afterwards.

Problems

1) Show that the set of initial conditions of the N-body problem which lead to a collision between a specific pair of particles has codimension 2 (i.e. 2 less than the dimension of the set of all initial conditions).
2) Verify that $x = -q(1-\sigma^2)$, $y = 2q\sigma$ is a solution of Box 2, Eq.(1), provided that $q \neq 0$ and

$$\sigma + \frac{1}{3}\sigma^3 = \frac{t}{\sqrt{2q^3}}.$$

Write down the appropriate collision solution, corresponding to $q = 0$, and show that the orbit varies smoothly with q as one passes through the collision orbit. Determine the geometric nature of the orbits. By treating the x,y plane as a complex z- plane and applying the Levi Civita transformation $\zeta = \sqrt{z}$, determine the geometric nature of the transformed orbits, and verify that they vary smoothly through the collision orbit.

Repeat this problem with the solution given in Box 2, Eq.(2), by varying e and keeping a fixed.

3) Two particles of mass m move along the x-axis, and are located at $x_1 = \sin^2(E/2)$ and $x_2 = -x_1$ at time $t = E - \sin E$. Verify that their motion satisfies the two-body equations if $G = 1$ and $m = 1/2$.

A third particle moves on the y-axis, and is massless. Show that its equation of motion may be written as the system

$$\begin{aligned}\dot{y} &= v\\ \dot{v} &= -\frac{y}{(y^2 + x_1^2)^{3/2}}.\end{aligned}$$

Show that there are three possible values of the constant c such that this system has the solution $y = cx_1$.

Change the independent variable in the system to E and code the equations numerically. By taking initial conditions close to one of the non-zero special solutions found previously, show that it is possible for the particle to be ejected with very high speed. (For a comparable problem with equal masses see Szebehely 1974.)