LECTURE 1

1. Introduction

In these lecture series we will discuss ancient solutions to parabolic equations and in particular to geometric flows.

By a geometric flow we typically mean a non-linear parabolic equation involving the change of a geometric quantity (often the curvature) on a manifold.

The simplest parabolic equation is the heat equation which describes the distribution of temperature. In fact, the temperature $u(x,t)$ at the point $x \in \mathbb{R}^n$ at time $t > 0$, then it satisfies the equation

$$u_t = \Delta u$$

where $\Delta u = \sum_{i=1}^n u_{x_i x_i}$.

Two of the remarkable properties of the heat equation are: the smoothing effect and infinite speed of propagation. Also, another very fundamental property is the parabolic scaling: if u solves the heat equation, so does

$$\hat{u} = \beta\, u(\alpha x, \alpha^2 t)$$

for any $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. All these can be seen from its fundamental solution (also a self-similar solution)

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}, \qquad t > 0$$

which has an initial data the dirac mass $\delta_0(x)$ at the origin.

It is easy to see that $\bar{u}(x,t) := u(x,-t)$ doesn't satisfy the same same equation that $u(x,t)$ but rather the backward heat equation

$$\bar{u}_t = -\Delta \bar{u}$$

This is in contrast to other time dependent equations such as the wave equation

$$u_{tt} = \Delta u$$

where $\bar{u}$ satisfies the same equation as u, i.e. time can be reversed.

Definition 1.1. *A solution $u(\cdot,t)$ to a parabolic equation is called ancient if it is defined for all time $-\infty < t < T$, $T \leq +\infty$.*
In the special case where the solution $u(\cdot,t)$ to a parabolic equation is defined for all $-\infty < t < +\infty$, it is called eternal.

Remark 1.2. Since time is cannot not reversed in a parabolic equation one expects that ancient solutions are very special and that one may expect to be able to provide a classification of such solutions under certain conditions. Note than in the case of the wave equation ancient is equivalent to globally defined, hence one doesn't expect that ancient solutions are particularly special !

We will see that ancient solutions arise as blow up limits at a singularities, hence their classification is important to understand the singularities of a parabolic nonlinear equation, in particular to geometric flows.

Examples of ancient solutions for the heat equation:

i. If $u(x,t) = U(x)$, $-\infty < t < +\infty$, where $U(x), x \in \mathbb{R}^n$ is harmonic, then $u(x,t)$ is in particular ancient (actually eternal).
ii. Let $u(x,t) = e^{x_1+t}$, $x = (x_1, \cdots, x_n)$, $-\infty < t < +\infty$ is ancient (actually eternal). This is a traveling wave solution.
iii. Let

$$u(x,t) = (4\pi(T-t))^{-n/2}\, e^{\frac{|x|^2}{4(T-t)}}, \qquad -\infty < t < T.$$

Then, $u(x,t)$ is an ancient solution which is defined up T.

1.1. **Outline of lectures.** In this series of lectures I will discuss classification results for ancient solutions to parabolic equations and geometric flows. The outline of my lectures is as follows:

Lecture 1: Liouville type theorems for ancient solutions to the heat equation on complete non-compact manifolds with nonnegative Ricci curvature.

Lecture 2: Classification of ancient solutions to seminilear heat equation related to the blow up analysis of the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^n$, for exponents $1 < p < \frac{n-2}{n+2}$,.

Lecture 3: Ancient compact solutions to curve shortening flow and mean curvature flow.

Lecture 4: Ancient compact solutions to the 2-dimensional Ricci flow.

2. Liouville's theorem for the heat equation

Let M be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}\,(M) \geq 0$. Here and later $\text{Ricci}\,(M)$ is the Ricci curvature and a manifold is complete if every geodesic extends to infinity.

Theorem 2.1 (Yau 1975)**.** *Any positive harmonic function u on M must be constant.*

In the special case where $M = \mathbb{R}^n$, the above result reduces to the classical Liouville's Theorem for harmonic functions on $\mathbb{R}^n$.

This theorem follows from the following *pointwise inequality* due to Cheng-Yau:

Theorem 2.2 (Cheng-Yau 1975)**.** *Let M be a complete manifold of dimension $n \geq 2$ with Ricci$(M) \geq 0$. Suppose that u is any positive harmonic function in a geodesic ball $B_R(x_0) \subset M$. Then, the inequality*

eqn-dh1

$$\frac{|\nabla u|}{u} \leq \frac{C_n}{R} \tag{2.1}$$

holds in $B_R(x_0)$, where C_n depends only on the dimension n.

Remark 2.3. In the case where $M^n = \mathbb{R}^n$ this resembles the derivative estimates for Harmonic functions, namely that for any harmonic function on $\mathbb{R}^n$, we have

$$\sup_{B_{\frac{R}{2}}(x_0)} |Du| \leq \frac{C_n}{R} \sup_{B_R(x_0)} |u|$$

for any $x_0 \in \mathbb{R}^n$ and $R > 0$. The significance of the Cheng-Yau estimate is that it is pointwise, but on the other hand it assumes that $u > 0$.

Motivated by the elliptic result one may ask:

Question: Does the analogue of Yau's theorem hold for positive solutions of the heat equation

$$u_t = \Delta u \qquad \text{on } M^n?$$

Answer: No. Example $u(x,t) = e^{x_1+t}$, $x = (x_1, \cdots, x_n)$ on $M^n := \mathbb{R}^n$.

However Souplet-Zang in [3] showed the following Liouville's theorem for harmonic functions on complete manifolds. We refer to the references in [3] for related work, an in particular with regard to the differentiable Harnack inequalities which will be discussed below.

thm-lhe

Theorem 2.4. *Let M^n be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with Ricci$(M^n) \geq 0$.*

(1) *If u be a positive ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that*
$$u(p,t) = e^{o(d(p)+\sqrt{|t|})} \qquad \text{as } d(p) \to \infty$$
then u is a constant.

(2) *If u be an ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that*
$$u(p,t) = o(d(p) + \sqrt{|t|}) \qquad \text{as } d(p) \to \infty$$
then u is a constant.

Remark 2.5. The example $u(x,t) = e^{x_1+t}$ on $\mathbb{R}^n$ shows that this theorem is sharp !

3. Li-Yau type inequalities

The parabolic analogue of the Chen-Yau estimate for solutions to the heat equation on complete non-compact manifolds was shown by Peter Li and S.T. Yau [2]. Another improved Harnack was later shown by R. Hamilton [1] for the compact case. Both of these inequalities have played a fundamental role to the development of geometric flows. Souplet-Zang in [3] localized the estimate by R. Hamilton on complete non-compact manifolds. In fact he proved the following result. Here and in the following we denote by $Q_{R,T} := B_R(x_0) \times [t_0 - T, t_0]$ and $Q_R := Q_{R,R^2}$.

thm-SZ1 **Theorem 3.1.** *Let M be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with Ricci$(M) \geq 0$. Suppose that u is any positive solution in $Q_{R,T}$ and that $u \leq L$ in $Q_{R,T}$. Then,*

eqn-dhhe3
$$\frac{|\nabla u|}{u} \leq C_n \left(\frac{1}{R} + \frac{1}{\sqrt{T}}\right) \left(1 + \ln \frac{L}{u}\right) \tag{3.1}$$

holds on $Q_{R/2,T/2}$, where C_n depends only on the dimension n.

Remark 3.2. Note that (3.1) is scaling invariant under parabolic scaling $\beta\, u(ax, a^2 t)$.

We will see next that can be directly used to give us the proof of the Liouville type theorem for the heat equation, Theorem (2.4).

Sketch of proof. The proof goes by the maximum principle. Suppose that $u \leq L$ in $Q_{R,T}$. Since our estimate is scaling invariant we may assume that $L = 1$, i.e. $0 < u \leq 1$. We set

$$f := \ln u, \qquad w := |\nabla(1-f)|^2 = \frac{|\nabla f|^2}{(1-f)^2}.$$

Then, the above estimate (after squaring and setting $L = 1$) becomes equivalent to

$$w \leq C_n \left(\frac{1}{R^2} + \frac{1}{T}\right)$$

which they show by the maximum principle !!

To this end, as usual we compute the evolution of w and after many standard calculations you show that

$$w_t - \Delta w \leq -\frac{2f}{1-f} \nabla f \cdot \nabla w - 2(1-f)\, w^2.$$

The details of this calculation in the simpler case where $M = \mathbb{R}^n$ will be given by Robin during the discussion session !!

□

Remark 3.3. The assumption that $u > 0$ is essential for this estimate to hold. Also, we will see that in the proof one works with $f := \ln u$ and the positivity of u is heavily used.

Remark 3.4. In the estimate differential Harnack inequality (3.1) above as well as in the Cheng-Yau estimate the assumption $\text{Ricci}\,(M) \geq 0$ can be generalized to the case $\text{Ricci}\,(M) \geq -k$, for some $k > 0$, with a slight modification of the statement. However, the assumption $\text{Ricci}\,(M) \geq 0$ is necessary for the Liouville theorems to hold both in the elliptic and parabolic cases !!

4. Proof of Theorem 2.4

Lets assume that $M = \mathbb{R}^n$, although the proof doesn't change at all if M is any complete, noncompact manifold with nonnegative Ricci curvature.

(1) Assume that u be a positive ancient solution to the heat equation on $\mathbb{R}^n \times (-\infty, T)$ such that

$$u(x,t) = e^{o(|x|+\sqrt{|t|})} \qquad \text{as } |x| \to \infty.$$

We want to see then u is a constant. To this end, we will apply (3.1) to $u+1$ instead of u. Fix a point $x_0 \in \mathbb{R}^n$ and a time t_0 and consider the cube $Q_R := B_R(x_0) \times [t_0 - R^2, t_0]$. Then, by our assumption we have

$$0 \leq \ln(u+1) \leq o(|x| + \sqrt{|t|}) = o(R), \qquad \text{on } Q_R,\ R \gg 1.$$

Hence, $u + 1 \leq e^{o(R)} := L_R$ and by (3.1), we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)+1} \leq \frac{C_n}{R}\left(1 + \ln\frac{L_R}{u+1}\right) \leq \frac{C(n,L)}{R}\big(1 + o(R)\big)$$

where we have used that

$$\ln\frac{L_R}{u+1} = \ln L_R - \ln(u+1) \leq \ln L_R = o(R).$$

Letting, $R \to +\infty$ we conclude that $\nabla u(0,t_0) = 0$, for all points (x_0,t_0), i.e. $u \equiv C$.

(2) Assume now that u is an ancient solution to the heat equation on $\mathbb{R}^n \times (-\infty, T)$ such that

$$u(p,t) = o(|x| + \sqrt{|t|}) \qquad \text{as } |x| \to \infty.$$

Fix a point $x_0 \in \mathbb{R}^n$ and a time t_0 and consider and let $A_R := \sup_{Q_R} |u|$, where $Q_R := B_R(x_0) \times [t_0 - R^2, t_0]$ as before. Our assumption implies that

$$A_R = o(R), \qquad \text{as } R \to +\infty.$$

Set

$$U := u + 2A_R.$$

Then,

$$A_R \leq U \leq 3A_R, \qquad \text{on } Q_R.$$

Apply (3.1) on Q_R and use that $U \geq A_R$ and $L_R = 3A_R$ to obtain

$$\frac{|\nabla U(x_0,t_0)|}{U(x_0,t_0)} \leq \frac{C}{R}\big(1 + \ln \frac{L_R}{U(x_0,t_0)}\big) \leq \frac{C}{R}\big(1 + \ln 3\big) \leq \frac{3C}{R}.$$

Here we have used that

$$\frac{U(x_0,t_0)}{R} \leq \frac{3A_R}{A_R} \leq 3 \quad \text{and} \quad \ln 3 \leq 3.$$

Thus,

$$\frac{|\nabla u(x_0,t_0)|}{U(x_0,t_0)} \leq \frac{3C}{R}$$

implying that

$$|\nabla u(x_0,t_0)| \leq 3C\,\frac{U(x_0,t_0)}{R} \leq 3C\,\frac{3A_R}{R} = \frac{o(R)}{R} \to 0, \quad \text{as } R \to +\infty.$$

We conclude that

$$\nabla u \equiv 0$$

i.e. $u \equiv$ constant.

References

Ha1 [1] Hamilton, R., *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993) 113–126.

LY [2] Li, P., and Yau, S. T., *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201.

SZ [3] Souplet, P. and Zhang, Q., *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds* Bull. London Math. Soc. **38** (2006) 1045–1053.

Yau [4] Yau, S. T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975) 201–228.