LECTURE 2

1. Semilinear Heat Equation

Consider next the semilinear heat equation

$$(\star_{SL}) \quad u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0,T)$$

in the subcritical range of exponents $1 < p < \frac{n+2}{n-2}$. For simplicity we will only consider nonnegative solutions $u \geq 0$.

It provides a prototype for the blow up analysis of geometric flows.

In particular in neckpinches of solutions to the Ricci flow and Mean Curvature flow.

Also in the characterization of rescaled limits as $t \to -\infty$ of ancient solutions.

We will next discuss parabolic Liouville type results related to the blow up analysis of solutions $u > 0$ to the semi-linear heat equation

$$(\star_{SL}) \quad u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0,T)$$

in the subcritical range of exponents $1 < p < \frac{n+2}{n-2}$.

Definition 1.1. *Assume that $u(\cdot,t) \in H^1(\mathbb{R}^n)$. We say that the solution u of $(\star_{SL})$ blows up in finite time T if*

$$\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty} = +\infty.$$

Since $p > 1$ the solution of the ODE

$$\frac{du}{dt} = u^p \iff u^{-p}\,du = dt \iff \frac{d(u^{1-p})}{1-p} = dt \iff u^{1-p} = T-(p-1)t$$

implies that

$$u(t) = \Big(\frac{1}{T-(p-1)t}\Big)^{1/(p-1)}$$

where $T = u^{1-p}(0)$. Hence, u wants to blow up in finite time.

The equation obeys the following scaling: is u solves $(\star_{SL})$, then for any $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta > 0$, the function

$$\hat{u}(x,t) = \beta\, u(\alpha x, \alpha^2 t)$$

is a solution of the same equation if

$$\hat{u}_t = \Delta\hat{u} + \hat{u}^p \iff \alpha^2\beta\, u_t = \alpha^2\beta\,\Delta u + \beta^p u^p \iff \alpha^2\beta = \beta^p$$

from which we conclude that $\beta = \alpha^{\frac{2}{p-1}}$, i.e.

$$u \text{ solves } (\star_{SL}) \quad \Longleftrightarrow \quad \hat{u}(x,t) = \alpha^{\frac{2}{p-1}} u(\alpha x, \alpha^2 t) \text{ solves } (\star_{SL}).$$

1.1. **The rescaled semi-linear heat equation.** A point (a, T) such that $\lim_{(x,t)\to(a,T)} u(x,t) = +\infty$ is called a singularity point.

Motivated by the scaling properties of the equation which we observed above, at a singularity point (a, T) we introduce the following re-scaling:

Self-similar scaling at a singularity at (a, T):

$$w(y,\tau) = (T-t)^{\frac{1}{p-1}}\, u(x,t), \;\; y = \frac{x-a}{\sqrt{T-t}}, \;\; \tau = -\log(T-t).$$

Theorem 1.2 (Giga - Kohn 1985)**.** $\|w(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq C$, $\tau > -\log T$.

The rescaled solution satisfies the equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

Objective: To analyze the blow up behavior of u one needs to understand the long time behavior of w as $\tau \to +\infty$.

This is closely related to the classification of bounded eternal solutions of $(\star)$.

1.2. **Eternal solutions of the semi-linear heat equation.** Related to the classification of singularities we pose the following problem:

Problem: Provide the classification of bounded positive eternal solutions w of equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

Eternal means that $w(\cdot,\tau)$ is defined for $\tau \in (-\infty, +\infty)$.

Theorem 1.3 (Giga-Kohn)**.** *The only bounded steady states of $(\star)$ are the constants:*

$$w = 0 \quad or \quad w = \kappa, \quad with \;\; \kappa := (p-1)^{-\frac{1}{(p-1)}}.$$

Theorem 1.4 (Giga - Kohn '87)**.** $\lim_{\tau\to\pm\infty} w(\cdot,\tau) =$ *steady state.*

The proof of the theorem above heavily uses the monotonicity of the following Lyapunov functional:

$$E(w) = \frac{1}{2}\int |\nabla w|^2 d\mu + \frac{1}{2(p-1)}\int w^2 d\mu - \frac{1}{p+1}\int w^{p+1} d\mu$$

where $d\mu = (4\pi)^{-\frac{n}{2}}\, e^{-\frac{|y|^2}{4}}\, dy$.

Theorem 1.5 (Giga - Kohn '87)**.** *For $p > 1$, the following holds:*

$$\frac{d}{d\tau} E(w(\tau)) = -\int w_\tau^2 \, d\mu.$$

Limits as $\tau \to \pm\infty$: Using the above Lyapunov functional one shows:

Theorem 1.6 (Giga - Kohn '87)**.** *The limits $w_{\pm\infty} := \lim_{s\to\pm\infty} w(\cdot, s)$ are steady states of $(\star)$. Hence $w_{\pm\infty} = 0, \kappa$.*
Since $E(w(s))$ is decreasing, there are only three possibilities:

$$w_{\pm\infty} = 0 \quad \text{or} \quad w_{\pm\infty} = \kappa \quad \text{or} \quad w_{-\infty} = \kappa \;\text{ and }\; w_{+\infty} = 0.$$

1.3. **Classification of Eternal solutions.** The following result holds regarding the classification of eternal solutions to $(\star)$.

Theorem 1.7 (Giga - Kohn '87 and Merle - Zaag '98)**.** *If w is bounded positive eternal solution of $(\star)$ defined on $\mathbb{R}^n \times (-\infty, +\infty)$, then*

$$w = 0 \;\text{ or }\; w = \kappa \;\text{ or }\; w = \phi(\tau - \tau_0).$$

Easy case - Giga-Kohn '89: If $w_{\pm\infty} = 0$ or $w_{\pm\infty} = \kappa$, then by the monotonicity of the energy $E(w(s))$ one concludes that $w \equiv 0$ or $w \equiv \kappa$.

Difficult case - (Merle - Zaag): Classify the orbits $w(\cdot, \tau)$ that connect the two steady states:

$$\lim_{\tau\to-\infty} w(\cdot, \tau) = \kappa \quad \text{and} \quad \lim_{\tau\to+\infty} w(\cdot, \tau) = 0.$$

1.4. **The result by Merle-Zaag.** Merle and Zaag proved the following classification theorem.

Theorem 1.8 (Merle - Zaag 1998)**.** *If $w(\cdot,\tau)$, $-\infty \le \tau \le +\infty$ is a bounded eternal solution of $(\star)$ such that $\lim_{\tau\to-\infty} w(\cdot,\tau) = \kappa$ and $\lim_{\tau\to+\infty} w(\cdot,\tau) = 0$, then $w(\cdot,\tau)$ is constant in space, hence the solution of the ODE*

$$w_\tau = -\frac{w}{p-1} + w^p \quad \tau \in \mathbb{R}$$

connecting the two steady states $w_{-\infty} = \kappa$ and $w_{+\infty} = 0$. It follows that $w(\tau) = \phi(\tau - \tau_0)$ with $\phi(\tau) = \kappa\,(1+e^\tau)^{-\frac{1}{(p-1)}}$.

Idea of the Proof: We will next discuss the main idea due to Merle-Zaag in the proof of the classification Theorem. They linearize the equation around the constant $w_{-\infty} = \kappa$, that is set $v := w - \kappa$. Then,

$$v_\tau = \mathcal{L}v + f(v)$$

where $\mathcal{L}$ is the linearized operator around the constant κ and is give by

$$\mathcal{L} := \Delta v - \frac{y}{2}\cdot\nabla v + v.$$

The error term is given by

$$f(v) := (v+\kappa)^p - \kappa^p - p\,\kappa^{p-1}\,v$$

and it is small as $v \to 0$.
The operator $\mathcal{L}$ is a well studied operator with known spectral properties, summarized in the next section.

1.5. **The linear operator $\mathcal{L}$.** The operator $\mathcal{L}$ is self adjoint in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}, e^{-|y|^2/4}dy)$. We introduce the norm and the inner product on $\mathfrak{H}$ by

$$\|f\|_\rho^2 = \int_{\mathbb{R}} f(y)^2 e^{-|y|^2/4}\,dy, \qquad \langle f, g\rangle_\rho = \int_{\mathbb{R}} f(y)g(y)e^{-|y|^2/4}\,dy.$$

Spectrum of $\mathcal{L}$:

$$\operatorname{spec}\mathcal{L} = \big\{1 - \frac{m}{2} \,|\, m \in \mathbb{N}\big\}.$$

In particular we have the following for the eigenvalues of $\mathcal{L}$:

- Positive: $\lambda_2 = 1$ (multiplicity 1- eigenfunction $\varphi_2 \equiv 1$),
  $\lambda_1 = 1/2$ (multiplicity n).
- Zero: $\lambda_0 = 0$ (multiplicity $\frac{n(n+1)}{2}$).
- Negative: Countable many $\lambda_{-1}, \lambda_{-2}, \cdots$

Eigenfunctions of $\mathcal{L}$: They are derived from Hermite polynomials.

1.6. Dominant mode as $\tau \to -\infty$.

Merle-Zaag decompose the function $v(\cdot,\tau)$ for $\tau \ll -1$ onto the eigenspaces of $\mathcal{L}$, that is they express

$$v = V_2^+ + V_1^+ + V_0 + V^-$$

where

- V_2^+ is the component corresponding to eigenvalue $\lambda_2 = 1$
- V_1^+ is the component corresponding to eigenvalue $\lambda_1 = 1/2$
- V^0 is the component corresponding to eigenvalue $\lambda_0 = 0$
- V_2^- is the component corresponding to negative (stable) eigenvalues $\lambda_{-1}, \lambda_{-2}, \cdots$

They show that: as $\tau \to -\infty$ one of the positive modes or the zero mode prevail.

Proposition 1.9 (Merle - Zaag 1998)**.** *As $\tau \to -\infty$, one of the three holds:*

(i) $|V_2^+(\tau)| + \|V_1^+(\tau)\|_{L^2_\rho} + \|V^-(\tau)\|_{L^2_\rho} = o(\|V^0(\tau)\|_{L^2_\rho})$,

(ii) $|V_2^+(\tau)| + \|V^0(\tau)\|_{L^2_\rho} + \|V^-(\tau)\|_{L^2_\rho} = o(\|V_1^+(\tau)\|_{L^2_\rho})$,

(iii) $\|V_1^+(\tau)\|_{L^2_\rho} + \|V_0\|_{L^2_\rho}(\tau) + \|V^-(\tau)\|_{L^2_\rho} = o(|V_2^+(\tau)|)$

i.e. as $\tau \to -\infty$ one of the positive modes or the zero mode prevail.

1.7. The ODE lemma.

A key ingredient in showing this proposition is the following ODE Lemma:

Lemma 1.10 (ODE Lemma)**.** *Let $X_0(\tau)$, $X_-(\tau)$ and $X_+(\tau)$ be absolutely continuous, real-valued functions that are nonnegative and satisfy*

- $(X_0, X_-, X_+)(\tau) \to 0$ *as* $\tau \to -\infty$
- $X_0(\tau) + X_-(\tau) + X_+(\tau) \neq 0$ *for all* $\tau \leq \tau_0$
- $\forall \epsilon > 0$, $\exists \tau_\epsilon \in \mathbb{R}$ *such that* $\forall \tau \leq \tau_\epsilon$, *and*

$$\left.\begin{aligned} \dot{X}_+ &\geq c_0 X_+ - \epsilon(X_0 + X_-) \\ |\dot{X}_0| &\leq \epsilon(X_0 + X_- + X_+) \\ \dot{X}_- &\leq -c_0 X_- + \epsilon(X_0 + X_+) \end{aligned}\right\} \tag{1.1}$$

Then either $X_0 + X_- = o(X_+)$ or $X_- + X_+ = o(X_0)$ as $\tau \to -\infty$.

Remark: You apply the ODE lemma on $V^+(\tau), V^0(\tau), V^-(\tau)$. To do so you need to show that the inequalities in the above lemma are satisfied by $V^+(\tau), V^0(\tau), V^-(\tau)$. This follows by the spectral properties of $\mathcal{L}$ and energy estimates on the equation

$$v_\tau = \mathcal{L}v + f(v)$$

where $\mathcal{L} := \Delta v - \frac{y}{2} \cdot \nabla v + v$ and the error term is given by

$$f(v) := (v+\kappa)^p - \kappa^p - p\,\kappa^{p-1}\,v.$$

1.8. **Dominant mode as $\tau \to -\infty$.**

Proposition 1.11 (Merle - Zaag 1998)**.** *As $\tau \to -\infty$, we have*

$$|V_1^+(\tau)| + \|V_0\|_{L^2_\rho}(\tau) + \|V^-(\tau)\|_{L^2_\rho} = o(\|V_2^+(\tau)\|_{L^2_\rho})$$

where we recall that $V_2^+(\cdot,\tau)$ is the projection of the solution $v(\cdot,\tau)$ to the eigenspace $\langle \varphi_2(y)\rangle$ corresponding to the eigenvalue $\lambda_2 = 1$, where $\varphi(y) \equiv 1.$. This means that

$$v(\cdot,\tau) = V_2^+(\tau) + \text{l.o.t.} = a(\tau)\cdot\varphi_2(y) + \text{l.o.t.} = a(\tau) + \text{l.o.t.}.$$

Note that: $V_2^+(\tau)$ is independent of the spatial variable !!

1.9. **The conclusion of the main result.** Using the last proposition namely that

$$v(\cdot,\tau) = a(\tau)\cdot 1 + l.o.t.$$

Merle-Zaag then show that

$$v(\cdot,\tau) \equiv V_2^+(\tau) \equiv a(\tau).$$

i.e. $v(\cdot,\tau)$ is independent of the spatial variable y.

Proposition 1.12. *The function $v(\cdot,\tau)$ is independent of the spatial variable y, i.e.*

$$v(\cdot,\tau) \equiv a(\tau), \qquad \forall\tau.$$

Hence, the same holds for $w := v + \kappa$.

They deduce that $w(\cdot,\tau) = \varphi(\tau)$, i.e. a solution of the ODE

$$w_\tau = -\frac{w}{p-1} + w^p \quad \tau \in \mathbb{R}.$$

Solving this ODE gives $w(\tau) = \kappa\,(1 + C\,e^\tau)^{-\frac{1}{(p-1)}}$ hence

$$w(\tau) = \phi(\tau - \tau_0), \qquad \text{with} \quad \phi(\tau) = \kappa\,(1+e^\tau)^{-\frac{1}{(p-1)}}.$$

Note: Their proof is rather complicated !!!

Jointly with Sigurd Angenent and Natasa Sesum we have recently applied similar techniques to geometric flows, in particular to the n-dimensional Mean curvature flow and 3-dimensional Ricci flow.