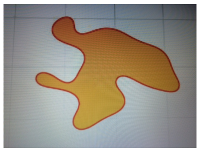
CURVE SHORTENING FLOW

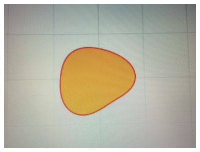
1. Introduction

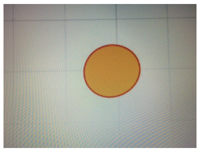
We consider an ancient embedded solution $F : \Gamma_t \to \mathbb{R}^2$ of the Curve shortening flow (CSF)

$$\frac{\partial F}{\partial t} = -\kappa\,\nu \tag{1.1}$$

with κ the curvature of the curve and ν the outer normal.







Contracting circles: Circles have constant curvature and they contract by constant speed. The equation says that if $r(r)$ denotes the radius of the circle, then

$$\frac{dr(t)}{dt} = -\kappa(t) = -\frac{1}{r(t)}.$$

Solving this equation for $r(t)$ gives

$$r(t) = \sqrt{2(T-t)}, \qquad \text{where } r(0) = \sqrt{2T}.$$

Also,

$$\kappa(t) = \frac{1}{\sqrt{2(T-t)}}.$$

If you assume that the curve is written as a graph $y = f(x,t)$, then f satisfies the quasilinear equation

$$f_t = \frac{f_{xx}}{1+f_x^2}.$$

Grim Reaper Solution:

$$f(x,t) = t + \log\sec x, \qquad x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

1.1. **Results.** Gage and Hamilton [5] that if Γ_0 is a convex curve embedded in $\mathbb{R}^2$, then equation (1.1) shrinks Γ_t to a point. In addition, the curve remains convex and becomes asymptotically circular close to its extinction time.

Grayson: studied the evolution of non-convex embedded curves under (1.1). He proved that if Γ_0 is any embedded curve in $\mathbb{R}^2$, the solution Γ_t does not develop any singularities before it becomes strictly convex.

1.2. **Evolution equations.** The evolution for the curvature κ of Γ_t is given by

$$\kappa_t = \kappa_{ss} + \kappa^3 \tag{1.2}$$

which is a strictly parabolic equation.

Let θ be the angle between the tangent vector and the x axis. For convex curves we can use the angle θ as a parameter. It has been computed in [5] that

$$\kappa_t = \kappa^2\, \kappa_{\theta\theta} + \kappa^3. \tag{1.3}$$

It turns out that the evolution of the family Γ_t is completely described by the evolution (1.3) of the curvature κ. Notice that κ is 2π-periodic.

Gage and Hamilton observed that a positive 2π periodic function represents the curvature function of a simple closed strictly convex C^2 plane curve if and only if

$$\int_0^{2\pi} \frac{\cos\theta}{\kappa(\theta)}\, d\theta = \int_0^{2\pi} \frac{\sin\theta}{\kappa(\theta)}\, d\theta = 0. \tag{1.4}$$

Our goal is to study ancient compact and convex solutions of the CSF. We will assume from now on that Γ_t is an ancient solution of the curve shortening flow defined on $(-\infty, T)$. We will also assume that our extinction time $T = 0$.

It is natural to consider the pressure function

$$p := \kappa^2$$

which evolves by

$$p_t = p\, p_{\theta\theta} - \frac{1}{2}\, p_\theta^2 + 2\, p^2. \tag{1.5}$$

We have already seen that the ancient solution to (1.3) defined by

$$p(\theta,t)=\frac{1}{2(-t)}$$

corresponds to a family of contracting circles. This solution is of type I and a contracting self-similar solutions.

S. Angenent showed the existence of another example of ancient compact solutions which are given by

$$p(\theta,t)=\lambda\Big(\frac{1}{1-e^{2\lambda t}}-\sin^2(\theta+\gamma)\Big) \tag{1.6}$$

where $\lambda>0$ and γ is a fixed angle. They turn out to be type II ancient solutions.

In this lecture we prove the following classification of ancient convex solutions to the curve shortening flow.

Theorem 1.1 (D., Hamilton, Sesum)**.** *Let $p(\theta,t)=\kappa^2(\theta,t)$ be an ancient solution to (1.5) defining an ancient convex solution to CSF. Then,*

i. *either $p(\theta,t)=\frac{1}{(-2t)}$, which corresponds to contracting circles, or*
ii. *$p(\theta,t)=\lambda(\frac{1}{1-e^{2\lambda t}}-\sin^2(\theta+\gamma))$, for two parameters $\lambda>0$ and γ, which corresponds to the Angenent ovals.*

2. Monotonicity formula

We will prove the theorem by introducing a monotone functional along the flow. Denote by

$$\alpha(\theta,t):=p_\theta(\theta,t).$$

By using (1.5) it easily follows that

$$\alpha_t=p\,(\alpha_{\theta\theta}+4\,\alpha). \tag{2.1}$$

We introduce the functional

$$I(\alpha)=\int_0^{2\pi}(\alpha_\theta^2-4\alpha^2)\,d\theta.$$

The following lemma shows the monotonicity of $I(\alpha)$ in time.

Lemma 2.1. *$I(\alpha(t))$ is decreasing along the flow (2.1). Moreover,*

$$\frac{d}{dt}I(\alpha(t))=-2\int_0^{2\pi}\frac{\alpha_t^2}{p}\,d\theta.$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dt}I(\alpha(t)) &= \int_0^{2\pi}(2\alpha_\theta\alpha_{\theta t} - 8\alpha\alpha_t)\,d\theta \\
&= -\int_0^{2\pi} 2\alpha_{\theta\theta}\alpha_t\,d\theta - 8\int_0^{2\pi}\alpha\alpha_t\,d\theta \\
&= \int_0^{2\pi}\frac{2(\alpha_t - 4\alpha p)\alpha_t}{p}\,d\theta - 8\int_0^{2\pi}\alpha\alpha_t\,d\theta \quad (\alpha_t = p(\alpha_{\theta\theta}+4\alpha) \\
&= -2\int_0^{2\pi}\frac{\alpha_t^2}{p}\,d\theta.
\end{aligned}$$

□

An easy computation shows that $I(\alpha(t)) \equiv 0$ on both the circles and the Angenent ovals which motivates the following:

3. Key step and proof of the Theorem

Proposition 3.1 (Key step)**.** *For any ancient convex solution to (1.5), we have*

$$I(\alpha(t)) \equiv 0, \qquad \text{for all } t \in (-\infty, 0).$$

Proof of Theorem 1.1. By Proposition 3.1 we have

$$I(\alpha(t)) \equiv 0, \qquad \text{for all } t < 0.$$

Lemma 2.1 implies that $\alpha_t \equiv 0$, that is,

$$p\,(\alpha_{\theta\theta} + 4\alpha) = 0$$

which means (since $p > 0$) that

$$\alpha_{\theta\theta} + 4\alpha = 0$$

and therefore

$$\alpha(\theta, t) = a_0(t)\cos 2(\theta + \gamma(t)) + b_0(t)\sin 2(\theta + \gamma(t))$$

for some functions in time $a_0(t)$, $b_0(t)$ and $\gamma(t)$.

Since $\alpha = p_\theta$, by integrating in θ we obtain

$$p(\theta, t) = a(t)\,\sin 2(\theta + \gamma(t)) + b(t)\,\cos 2(\theta + \gamma(t)) + c(t) \tag{3.1}$$

for $a(t) = \frac{a_0(t)}{2}$, $b(t) = -\frac{b_0(t)}{2}$ and another function in time $c(t)$.

In the case where $a(t) \equiv 0$ and $b(t) \equiv 0$, you have that $p(\theta, t) = c(t)$ and you recover the contracting spheres.

In the other case, if we plug $p(\theta,t)$ back to equation (1.5) and you deduce after some straight-forward calculations that

$$p(\theta,t) = \lambda(\frac{1}{1-e^{2\lambda t}} - \sin^2(\theta+\gamma)).$$

$\square$

4. Proof of the key step

We will now outline the proof of the key stop which says that

$$I(\alpha(t)) \equiv 0.$$

The proof will be given in two steps.

Step 1: $\liminf_{t\to-\infty} I(\alpha(t)) \le 0$.

Step 2: $\lim_{t\to 0} I(\alpha(t)) = 0$.

Once we know these two steps, the monotonicity

$$\frac{d}{dt} I(\alpha(t)) \le 0$$

will then readily imply that $I(\alpha(t)) \equiv 0$, for all $t < 0$, as desired.

The proof of Step 2 uses the Gage-Hamilton result says that as $t \to 0$ the solution becomes more and more circular.

Let us state a bit more explicitly result by Gage and Hamilton in [5]. The curvature satisfies

$$\kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3.$$

Consider the rescaled curvature $\tilde{\kappa}$ is defined by

$$\tilde{\kappa}(\theta,\tau) = k(\theta,t)\sqrt{-2t}, \qquad \text{with } \tau = -\frac{1}{2}\log(-t). \tag{4.1}$$

Theorem 4.1 (Gage, Hamilton)**.** *If Γ_0 is a closed convex curve embedded in the plane $\mathbb{R}^2$, the curve shortening flow shrinks Γ_t to a point in a circular manner. Moreover, for any $\eta \in (0,1)$ and $m \ge 1$ we have*

$$|\tilde{\kappa} - 1| \le C(\eta)\, e^{-2\eta\tau}, \qquad |\frac{\partial^m \tilde{\kappa}}{\partial \theta^m}| \le C_m(\eta)\, e^{-2\eta\tau} \tag{4.2}$$

for $\tau \gg 1$.

This means that as $\tau \to +\infty$, $\tilde{\kappa}_\theta \to 0$ and $\tilde{p}_\theta \to 0$, $\tilde{p} := \tilde{\kappa}^2$. By refining the Gage-Hamilton estimate we conclude that $\alpha := p_\theta$

$$I(\alpha) = \int_0^{2\pi} (\alpha_\theta^2 - 4\alpha^2)\, d\theta \equiv 0.$$

4.1. **Proof Step 1:** $\liminf_{t\to-\infty} I(\alpha(t)) \le 0$.

Proof. On an ancient solution we have $k_t \ge 0$ which gives $p_t \ge 0$. Hence, $p(\cdot,t) \le C < \infty$, for all $t < -1 < 0$.

If we differentiate (1.5) in θ we get

$$(p_\theta)_t = p(p_\theta)_{\theta\theta} + 4pp_\theta \tag{4.3}$$

which implies

$$\frac{1}{2p}\,(p_\theta^2)_t = p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2$$

and therefore

$$\left(\frac{p_\theta^2}{2p}\right)_t = \frac{(p_\theta^2)_t}{2p} - \frac{p_\theta^2\; p_t}{2p^2} \le p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2$$

where we used that $p_t \ge 0$. Integrating the above inequality gives

$$\frac{d}{dt}\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \le \int_0^{2\pi} \big(\,p_\theta(p_\theta)_{\theta\theta} + 4p_\theta^2\,\big)\,d\theta$$

and after integration by parts we get

$$\frac{d}{dt}\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \le \int_0^{2\pi} \big(-p_{\theta\theta}^2 + 4p_\theta^2\big)\,d\theta = -I(\alpha(t)). \tag{4.4}$$

On the other hand, from the inequality

$$p\,p_{\theta\theta} - \frac{1}{2}\,p_\theta^2 + 2\,p^2 = p_t \ge 0$$

dividing by p and integrating we obtain

$$\int_0^{2\pi} -\frac{1}{2}\frac{p_\theta^2}{p} + 2\,p\,d\theta \ge 0$$

or

$$\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \le 2\int_0^{2\pi} p\,d\theta \le C$$

since p is bounded for $t < -1 < 0$. Combining this with (4.4) implies that

$$\int_{-\infty}^{t_0} I(\alpha(t))\,dt \le \limsup_{t\to-\infty}\int_0^{2\pi} \frac{p_\theta^2}{2p}\,d\theta \le C.$$

This implies

$$\liminf_{t\to-\infty} I(\alpha(t)) \le 0.$$

Indeed, assume that

$$\liminf_{t\to-\infty} I(\alpha(t)) > 0.$$

Then, this means that $I(\alpha(t)) \geq c > 0$ for $t \leq t_0 \ll -1$. Thus,

$$\int_{-\infty}^{t_0} I(\alpha(t))\,dt \geq c \int_{-\infty}^{t_0} dt = +\infty$$

which contradicts the fact that this integral is bounded. □

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