1 Lebesgue point, Vitali covering

As I said, one of the themes we'll be dealing with a lot in this area of mathematics is that some nice properties hold for all but a set of small measure. So we first pretend everything is smooth and analyze the good set; then estimate the size of the set where things are not smooth. For the latter we often need to use smart covering scheme.

Theorem 1 (Vitali's covering theorem)**.** *Let $\mathcal{F}$ be a collection of closed balls in $\mathbb{R}^n$ with (uniform control on the diameters)*

$$\sup_{B\in\mathcal{F}} \operatorname{diam} B < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$, such that

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B_i\in\mathcal{G}} 5B_i,$$

where each $5B_i$ is a concentric ball of B_i, with five times the radius of B_i.

Remark 2. *The words to pay attention to here are countable and disjoint. Notice that the original collection $\mathcal{F}$ could contain uncountably many balls, and they do not need to be disjoint; and yet, if we're willing to cover by concentric balls five times big, we can find a better collection that is countable and pairwise disjoint. It is much easier to estimate the size of a countable union of disjoint balls, than an uncountable union of balls that have a lot of overlap!*

What we use more often is an immediate corollary of this theorem. We say a collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is a **cover** of a set $A \subset \mathbb{R}^n$, if

$$A \subset \bigcup_{B\in\mathcal{F}} B.$$

Fine cover of A, if in addition, for each $x \in A$

$$\inf_{\substack{B\ni x\\ B\in\mathcal{F}}} \operatorname{diam} B = 0,$$

that is, any point in A can be covered by balls in $\mathcal{F}$ of sufficiently small radius.

Corollary 3. *Assume $\mathcal{F}$ is a fine cover of A and*

$$\sup_{B\in\mathcal{F}} \operatorname{diam} B < \infty.$$

(By the previous theorem, we get that) Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$, such that

$$A \subset \bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

Moreover, for any finite subset $\{B_1, \cdots, B_k\} \subset \mathcal{F}$ (finite collection of balls in $\mathcal{F}$), we have

$$A \setminus \bigcup_{j=1}^{k} B_j \subset \bigcup_{B\in\mathcal{G}\setminus\{B_1,\cdots,B_k\}} 5B.$$

Not only we have a good covering of the entire set A, we also have good coverings of any portion of A, after removing finitely many balls.

Remark 4. *We make the additional assumption of fine cover here, so that for any $x \in A \setminus \bigcup_{j=1}^{k} B_j$, we can find a ball $B \in \mathcal{F}$ that contains x, and moreover its radius is sufficiently small so that B is disjoint from all of B_j's.*

Exercise Let $\delta > 0$ be arbitrarily small. Let $U \subset \mathbb{R}^n$ be an open set with $|U| < +\infty$. Show that there exists $\{B_i\}$, a countable collection of disjoint closed balls in U with $\operatorname{diam} B_i \leq \delta$, such that

$$\left| U \setminus \bigcup_i B_i \right| = 0.$$

Hint: Come up with a fine cover of U, and then use the corollary to Vitali's covering theorem.

2 Definitions & properties of Radon measures

Let $\mathcal{M} \subset \{A \subset \mathbb{R}^n\}$ be a *suitable* collection of sets in $\mathbb{R}^n$ (c.f. σ-algebra). A **measure** $\mu : \mathcal{M} \to [0, +\infty]$ is a function that assigns to each set a positive number (or infinity), which satisfies

- $\mu(\emptyset) = 0$ (the number assigned to the empty set is zero, very naturally);
- If $\{E_i\}$ is a countable collection of *disjoint sets* in $\mathcal{M}$, then $\mu\left(\bigcup_i E_i\right) = \sum_i \mu(E_i)$.

In particular,

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

if A and B are disjoint.

We say a measure μ on $\mathbb{R}^n$ is a **Radon measure** if it is a Borel regular measure (whose rigorous definition is not so important in this lecture) that is locally finite, i.e. $\mu(B) < +\infty$ for any closed ball $B \subset \mathbb{R}^n$. (Of course when we take balls of bigger and bigger radius, the number $\mu(B)$ can be huge; but every one of them is finite.)

Theorem 5 (Approximation theorem for Radon measures)**.** *For any $E \in \mathcal{M}$,*

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) : K \subset E, K \text{ is compact}\} \\ &= \inf\{\mu(U) : U \supset E, U \text{ is open}\}\end{aligned}$$

Alternatively, let $\epsilon > 0$ be arbitrarily small. For any set $E \in \mathcal{M}$ with $\mu(E) < +\infty$, there exist a compact set $K \subset E$ and an open set $U \supset E$ such that

$$\mu(E \setminus K), \mu(U \setminus E) < \epsilon.$$

Because $K \subset U$, and K is compact, U is open, there exists a smooth *bump function* f satisfying (f takes value between 0 and 1)

$$0 \leq f \leq 1, \quad f \equiv 1 \text{ on } K, \quad f \equiv 0 \text{ in the complement of } U.$$

Then

$$\mu(K) \leq \int f d\mu \leq \mu(U).$$

Since both $\mu(K)$ and $\mu(U)$ are very close to the measure of E, we have

$$\left| \int f d\mu - \mu(E) \right| \leq \epsilon.$$

The moral is: We can study a measure, by looking at how it behaves when we *test it against smooth functions* $\int f d\mu$.

3 Convergence & compactness of Radon measures

Let $\{\mu_j\}$ and μ be Radon measures on $\mathbb{R}^n$. We say μ_j **converges to μ in the weak sense**, and write $\mu_j \rightharpoonup \mu$, if for any $f \in C_c^0(\mathbb{R}^n)$ (smooth function with compact support)

$$\int f d\mu_j \to \int f d\mu. \tag{6}$$

(6) is a natural definition, because of this moral of the previous section that we study a measure by looking at the integral of smooth functions $\int f d\mu$.

Theorem 7 (Compactness for Radon measures)**.** *Suppose $\{\mu_j\}$ is a sequence of Radon measures on $\mathbb{R}^n$, such that for any compact set $K \subset \mathbb{R}^n$, there is a uniform upper bound on the $\mu_j(K)$'s*

$$\sup_j \mu_j(K) < +\infty.$$

Then there exist a Radon measure μ on $\mathbb{R}^n$ and a subsequence $\{\mu_{j_k}\} \subset \{\mu_j\}$, such that

$$\mu_{j_k} \rightharpoonup \mu \text{ as } k \to +\infty.$$

Proof. I will only say a few words on how to start the proof. The weak convergence means that for any $f \in C_c^0(\mathbb{R}^n)$

$$\int f d\mu_{j_k} \to \int f d\mu.$$

Fix any such function f, say it is supported in some ball B (i.e. $f \equiv 0$ outside of B)

$$\left|\int f d\mu_j\right| = \left|\int_B f d\mu_j\right| \le \max|f| \cdot \sup_j \mu_j(B) < +\infty.$$

$\left\{\int f d\mu_j\right\}$ is a sequence of real numbers that are uniformly bounded, so it has a convergent subsequence. By a diagonalisation argument, there is a subsequence such that

$$\lim_{k\to\infty} \int f d\mu_{j_k} \text{ exists for every } f \in C_c^\infty(\mathbb{R}^n).$$

We define a linear functional $\ell : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ as

$$\ell(f) := \lim_{k\to\infty} \int f d\mu_{j_k}.$$

By Reisz representation theorem, there is a Radon measure μ on $\mathbb{R}^n$ such that

$$\int f d\mu = \lim_{k\to\infty} \int f d\mu_{j_k}.$$

□

Property 8. *Let $\{\mu_j\}$ and μ be Radon measures on $\mathbb{R}^n$ such that $\mu_j \rightharpoonup \mu$. Then*

$$\mu(U) \le \liminf_{j\to\infty} \mu_j(U), \quad \text{if } U \text{ is an open set;}$$

$$\mu(K) \ge \limsup_{j\to\infty} \mu_j(K), \quad \text{if } L \text{ is a compact set.}$$

Moreover, if $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{j\to\infty} \mu_j(E).$$

4 Hausdorff measure

In $\mathbb{R}^3$, suppose there are three objects in front of you: a surface S, a line L, and a cluster of points P. When you measure their *sizes* by the Lebesgue measure, you get zero for all of them, even though geometrically they are completely different objects. The reason is that Lebesgue measure is inherently 3-dimensional; and these objects have different dimensions. We want a measure that tells these 3 objects apart and give their respective sizes honestly: surface area, curve length, count the number of points! This is what Hausdorff measure can do.

Given $0 \le d \le n$ (d may be a non-integer), the d-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$, denoted by $\mathcal{H}^d(A)$, is defined and computed as follows. We cover A by a countable union of balls $\{B_i\}$ (draw and write) with $\operatorname{diam} B_i \le \delta$, and we sum up the size of the balls by pretending these balls are d-dimensional $\sum_i \omega_d \left(\frac{\operatorname{diam} B_i}{2}\right)^d$. There are many different ways to cover A and we want to choose the most efficient cover. So we take the infimum of this number. We then take the limit as $\delta \to 0$.

$$\mathcal{H}^d(A) = \lim_{\delta \to 0} \inf \left\{ \sum_i \omega_d \left(\frac{\operatorname{diam} B_i}{2} \right)^d : A \subset \bigcup_i B_i, \operatorname{diam} B_i \le \delta \right\}.$$

We need to take $\delta \to 0$: think about a set A like this, it is a curve living inside a unit ball, but it winds around many times so its length is huge. Think about what would be $\mathcal{H}^1(A)$ if we do not insist on passing $\delta \to 0$ in the last step.

Examples: Let $n = 3$.

- When $d = n = 3$, the Hausdorff measure is exactly the Lebesgue measure of A (recall that we also define the Lebesgue measure of a set by filling it by balls or cubes)

$$\mathcal{H}^n(A) = |A|.$$

- Therefore $\mathcal{H}^3(S) = 0$; $\mathcal{H}^2(S)$ is exactly the surface area of S; $\mathcal{H}^1(S) = +\infty$. (You may compute it yourself assuming S is the square. For example consider how many balls of radius δ you need to cover the square.)
- $\mathcal{H}^3(L) = 0$, $\mathcal{H}^2(L) = 0$, $\mathcal{H}^1(L)$ is exactly the length of L.
- The surface measure σ we introduced yesterday in the review session, is also secretly a Hausdorff measure

$$\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}, \quad \text{i.e. } \sigma(A) = \mathcal{H}^{n-1}(A \cap \partial\Omega).$$

From the above examples, we realize that for a given set A, the Hausdorff measure only gives a meaningful number (a number that is not zero or infinity), if we're looking at the right dimension! In fact, there is always some d_0, such that

$$\mathcal{H}^d(A) = +\infty \text{ for all } d < d_0,$$

and

$$\mathcal{H}^d(A) = 0 \text{ for all } d > d_0.$$

Such d_0 is called the *Hausdorff dimension of the set.*