In today's preview session the goal is to cover the following topics. Firstly the divergence theorem and Green's theorem in smooth settings. Tatiana will talk about generalizations of these results for non-smooth domains, which will in turn be useful in for example geometric minimization problems. Secondly properties of Lipschitz functions.

1 Divergence theorem in smooth setting

We consider a C^1 domain $\Omega \subset \mathbb{R}^n$, which means a domain that is locally characterized by a C^1 graph. For example

$$\Omega = \{(\bar{x}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(\bar{x})\}.$$

Here $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is of class C^1, i.e. continuously differentiable; and the domain Ω is the region above the graph of ϕ. For simplicity, let's assume $\partial\Omega$ passes through the origin (modulo translation) and that the tangent plane to ϕ at the origin is flat and $\mathbb{R}^{n-1}$ (the plane spanned by all but the last coordinate axes) $\phi'(x_0) = 0$ (modulo rotation).

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function that is smooth and has compact support, that is, there exists a compact set K such that $u \equiv 0$ outside of K; in $\mathbb{R}^n$ this means the function u is trivial outside of some closed ball B (to make sure the integrals in consideration are all finite). The **Divergence theorem** says

$$\iint_\Omega \operatorname{div} u \, dx = \int_{\partial\Omega} u \cdot \nu \, d\sigma,$$

where

$$\operatorname{div} u(x) = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}(x),$$

$$\nu \text{ is the normal vector of } \Omega \text{ pointing outwards,}$$

$$\sigma \text{ measures the surface area of the boundary } \partial\Omega.$$

The physical interpretation of the divergence theorem is that the outward flow of a vector field through a surface $\partial\Omega$ equals the integral of its divergence on the enclosed region Ω. In the simplest 1D case, this is just the fundamental theorem of calculus:

$$\int_{[a,b]} u' dx = u(b) - u(a)$$

since $\partial[a, b] = \{a, b\}$ with opposite orientation.

Our next **Goal** is to write out ν and $d\sigma$ explicitly. Notice that

$$\partial\Omega = \{(\bar{x}, t) : t = \phi(\bar{x})\}, \text{i.e. the zero level set of the function } f(\bar{x}, t) := t - \phi(\bar{x}).$$

Along the level set, the function value stays the same; and the function value only changes in the normal direction. For this reason, the normal vector to the level set is given by its derivative $Df(\bar{x}, t) = (-D\phi(\bar{x}), 1)$, here the total gradient of ϕ is given by $D\phi = (\partial_{x_1} u, \cdots, \partial_{x_{n-1}} u)$. We normalize to get a normal vector of unit length: However this is the unit normal vector pointing inwards, because the last exponent is positive; so

$$\nu = \frac{(D\phi(\bar{x}), -1)}{\sqrt{1 + |D\phi(\bar{x})|^2}}.$$

For any ball B centered at the boundary, it is clear that if we look at the surface area of $B \cap \partial\Omega$, it should be greater or equal to the size of the projection of B onto the tangent space $\mathbb{R}^{n-1}$. In fact, the surface measure can be expressed in terms of the Lebesgue measure in $\mathbb{R}^{n-1}$:

$$d\sigma(\bar{x}, \phi(\bar{x})) = \sqrt{1 + |D\phi(\bar{x})|^2}\, d\bar{x}.$$

This is because the boundary is characterized by the map

$$\bar{x} \in \mathbb{R}^{n-1} \mapsto (\bar{x}, \phi(\bar{x})) \in \partial\Omega \subset \mathbb{R}^n.$$

We may compute the derivative of this map and use the so-called **Area formula** (which relates the size of a set $A \subset \mathbb{R}^{n-1}$ and the size of its image in $\mathbb{R}^n$, by the determinant of the derivative of this map), one gets

$$d\sigma(\bar{x}, \phi(\bar{x})) = \sqrt{1 + |D\phi(\bar{x})|^2}\, d\bar{x}.$$

Heuristically, if the variable downstairs changes by $d\bar{x}$, the corresponding height changes by $D\phi(\bar{x})d\bar{x}$.

1.1 Green's theorem

Now we use divergence theorem to show the **Green's theorem** for C^1 domains. Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be two smooth, compactly supported functions. Product rule shows that

$$\operatorname{div}(u\nabla v) = \sum \frac{\partial}{\partial x_j}(u\frac{\partial v}{\partial x_j}) = \sum \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j} + u\sum \frac{\partial^2 v}{\partial x_j^2} = \nabla u \cdot \nabla v + u\Delta v,$$

where $\Delta v = \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2}$ is called the Laplacian of v. By Divergence theorem, it follows

$$\begin{aligned} \iint_\Omega \nabla u \cdot \nabla v + \iint_\Omega u\,\Delta v &= \iint_\Omega \operatorname{div}(u\nabla v) \\ &= \int_{\partial\Omega} u\nabla v \cdot \nu\, d\sigma = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, d\sigma. \end{aligned}$$

Thus by exchanging the rolls of u and v, we get the Green's theorem

$$\iint_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\, d\sigma.$$

2 Topics on Lipschitz functions

We say a function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if there exists a uniform constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y| \text{ for every } x, y \in \text{ the domain of definition } A.$$

The smallest constant C satisfying the above is called the Lipschitz constant of f, denoted as $\operatorname{Lip}(f)$. For example, $f(x) = |x|$ is a Lipchitz function with constant 1.

Theorem 1 (Extension of Lipschitz functions)**.** *Assume $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is Lipschitz. Then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that*

- $\bar{f} = f$ *on* A*;*
- $\mathrm{Lip}(\bar{f}) \le \sqrt{m}\,\mathrm{Lip}(f)$.

Proof. Assume $m = 1$. Such a function $\bar{f}$ should satisfy

$$|\bar{f}(x) - \bar{f}(a)| \le C|x - a|,$$

for every $x \in \mathbb{R}^n$ and $a \in A$; thus

$$\bar{f}(x) \le f(a) + C|x - a|.$$

We define

$$\bar{f}(x) := \inf_{a \in A}\{f(a) + \mathrm{Lip}(f)|x - a|\}.$$

It's not difficult to show $\bar{f} = f$ on A.

A more elaborate proof can in fact show the existence of an extension satisfying $\mathrm{Lip}(\bar{f}) = \mathrm{Lip}(f)$. ☐

If we look at the example $f(x) = |x|$, we can see that this function is differentiable everywhere except at the origin. This is in fact true for all Lipschitz functions:

Theorem 2 (Rademacher's theorem)**.** *Lipschitz functions are differentiable almost everywhere.*

To make it rigorous, we need the following definition: The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $x \in \mathbb{R}^n$ if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0,$$

or equivalently

$$f(y) = f(x) + L(y - x) + o(|y - x|).$$

If such a map L exists, it is clearly unique. We write $Df(x)$ for L, and call it *the derivative of* f *at* x.

Theorem 3. *If* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is Lipschitz, then* f *is differentiable at almost every* $x \in \mathbb{R}^n$*. In other words, the set of* x *where* f *is not differentiable has Lebesgue measure zero, and so from a measure point of view, the bad set does not exists.*

For problems in non-smooth settings, statements like Rademacher's theorem is very common: some nice properties hold except for a set of small size. The philosophy then is: pretend everything is smooth and prove the desired results; estimate the size of the set where things are not so nice. It is the second aspect that demands smart covering schemes, such as Vitali's covering theorem, which we will cover shortly.

2.1 Exercise

Exercise: Because Lipschitz functions are differentiable almost everywhere, the divergence theorem we talked about before also holds for Lipschitz domains, that is, domains that are given by Lipschitz graphs rather than C^1 graphs. The explicit formula

$$d\sigma(\bar{x}, \phi(\bar{x})) = \sqrt{1 + |D\phi(\bar{x})|^2}\,d\bar{x}$$

also holds. (Here $D\phi(\bar{x})$ is no longer the classical gradient of ϕ, it is the linear mapping whose existence is guaranteed by Rademacher's theorem.) Show that if $\mathrm{Lip}(\phi) \leq \epsilon$, then for any $x \in \partial\Omega$,

$$\frac{\sigma(B_r(x))}{\omega_{n-1} r^{n-1}} \sim 1$$

$$1 - C_1\epsilon^2 \leq \frac{\sigma(B_r(x))}{\omega_{n-1} r^{n-1}} \leq 1 + C_2\epsilon^2.$$

The denominator is the Lebesgue measure of a ball of radius r in $\mathbb{R}^{n-1}$. In plain words, it says that if the boundary is flat (due to the smallness of the Lipschitz constant), then the surface measure is very close to the canonical one.