1 Isoperimetric inequality

Recall the isoperimetric inequality Tatiana showed you this morning:

Theorem 1.

$$n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \leq |\mu_E|(\mathbb{R}^n).$$

Equality holds if and only if E is a ball.

Proof. Queen Dido's story. For the proof of the isoperimetric inequality, let me translate it to a minimization problem equivalent to Queen Dido's.

Let m be a positive number. We fix the volume of sets to be m, then the inequality becomes to show

$$n\omega_n^{\frac{1}{n}}m^{\frac{n-1}{n}} \leq |\mu_E|(\mathbb{R}^n), \qquad \text{for all sets with (fixed volume) } |E| = m;$$

in other words, we want to look at the minimization problem

$$\min\{|\mu_E|(\mathbb{R}^n) : |E| = m\}$$

and show that $E = B$ (a ball of volume m) is the minimizer.

There are many different ways of proof. My favorite one uses a procedure called Steiner's symmetrization, and I will illustrate how it works in the plane. Suppose E is a set that minimizes (). We define another set F by the following procedure. Fix a line L. We look at a slicing of E (a slicing is $E \cap L'$ the intersection of E with a line L' orthogonal to L); we compute the length of the slicing, and draw a line segment centered on L with the same length as the length of the slicing. This way we construct a new set F, that is symmetric to the line L; and since the area (in the plane the Lebesgue measure is the area) is the sum of the lengths of all slicings, this construction does not change the area of E. I claim that this construction decreases the perimeter

$$|\mu_F|(\mathbb{R}^n) \leq |\mu_E|(\mathbb{R}^n)$$

(the proof of the claim requires some work and I can't do it here. But you can imagine: because this procedure merges different parts of the set together, some portion that is originally in the boundary of E and thus contributes to the perimeter of E disappear).

To recap what we just did: we can apply the above Steiner symmetrization to the minimizing set so that it becomes symmetric with respect to L and the perimeter decreases. Since we can apply the symmetrization to every line, it follows that the minimizer is symmetric in every direction, so it is a ball. □

2 Examples of nasty sets of finite perimeters

When you first learn sets of finite perimeters, it's helpful to think of C^1 domains for intuition; but sets of finite perimeters include some pathological ones.

Example 1. For any $\epsilon > 0$, we can find a set of finite perimeter (the perimeter is actually bounded above by 1) $E \subset B$, B is the unit ball in $\mathbb{R}^n$, such that

$$|E| < \epsilon, \text{ and yet } |\partial E| \geq |\operatorname{spt} \mu_E| > \omega_n - \epsilon.$$

On one hand, the size/volume of E is sufficiently small, and yet its boundary has volume sufficiently close to the volume of B. This is very suprising.

Let $\{x_i\}$ be the collection of rational points in B. Suppose $\{r_i\}$ is a sequence of real numbers (lying between 0 and ϵ) $0 < r_i < \epsilon$ satisfying $n\omega_n \sum_i r_i^{n-1} \leq 1$. Then one can check that

$$E = \bigcup_i B(x_i, r_i)$$

is a set of finite perimeter (whose perimeter is bounded above by 1) and it satisfies the above.

Example 2. Google four-corner Cantor set. If you know what Cantor set is in the real line, four-corner Cantor set is by similar construction but lives in the plane instead of the real line. There is a specific choice of parameter so that the Cantor set is a set of finite perimeter. But surprisingly, the reduced boundary is empty!

The message here is: sets of finite perimeters are in fact a very large class of sets, and there are some nasty ones; however, Tatiana will show that on the other hand, a perimeter minimizer, one that minimizes the perimeter, has to be a rather nice object.